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On Star Pre Open Sets and Strong Star Pre Open Sets in Ideal Bitopological Space

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المجموعات المفتوح Star Pre والمجموعات المفتوحة Star pre القوية في الفضاءات التبولوجية الثنائية المثالية

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Abstract:
Firstly, in this study, we introduce the concept of $\mathrm{Pre}^*_{i,j}\mathrm{I}$-open set in ideal bitopological spaces, which is weaker than the concept of $\mathrm{Pre}_{i,j}\mathrm{I}$-open set in ideal bitopological spaces. Subsequently, we present the notion of strong $\mathrm{Pre}^*_{i,j}\mathrm{I}$-open set in ideal bitopological spaces and discuss some of their properties. Furthermore, based on this new concept, we define the strong $\mathrm{Pre}^*_{i,j}\mathrm{I}$ -interior and strong $\mathrm{Pre}^*_{i,j}\mathrm{I}$- closure, along with an investigation of their various properties.

Keywords: Ideal bitopological space, $\mathrm{Pre}^*_{i,j}\mathrm{I}$-open set, strong $\mathrm{Pre}^*_{i,j}\mathrm{I}$-open set, $\mathrm{SP}^*_{i,j}\mathrm{I}\text{-}\mathrm{Int}(\mathcal{U})$,$\mathrm{SP}^*_{i,j}\mathrm{I}\text{-}\mathrm{Cl}(\mathcal{U})$.

الملخص:
في هذه الورقة، قدمنا واستكشفنا مفهوم المجموعات المفتوحة من النوع $\mathrm{Pre}^*_{i,j}\mathrm{I}$ والمجموعات المفتوحة القوية من النوع $\mathrm{Pre}^*_{i,j}\mathrm{I}$ضمن إطار الفضاءات الطوبولوجية الثنائية المثالية. تُسهم هذه المفاهيم في فهم أعمق للبنى الطوبولوجية من خلال تقديم أشكال معممة أضعف ولكنها غنية بالخصائص. كما طورنا المفاهيم المقابلة لمؤثرات الإغلاق والداخلية القوية من النوع$\mathrm{Pre}^*_{i,j}\mathrm{I}$ ودرسنا خصائصها الأساسية. تُمهد هذه النتائج الطريق لمزيد من الدراسات حول المجموعات المفتوحة المعممة وبديهيات الفصل في الفضاءات الطوبولوجية الثنائية والمثالية.

الكلمات المفتاحية: الفضاء التبولوجي الثنائي المثالي، 1لمجموعة المفتوحة من النوع $\mathrm{Pre}^*_{i,j}\mathrm{I}$ ، المجموعة المفتوحة القوية من النوع $\mathrm{Pre}^*_{i,j}\mathrm{I}$ ، الداخلية والغلاقة للنوع $\mathrm{SSP}^*_{i,j}\mathrm{I}\text{-}\mathrm{Int}(\mathcal{U})$,$\mathrm{SP}^*_{i,j}\mathrm{I}\text{-}\mathrm{Cl}(\mathcal{U})$.

Introduction:

The study of topological structures has evolved significantly through the integration of new abstract concepts that enhance classical topology. One such advancement is the bitopological space, first introduced by Kelly (1963) [1], where a set $\mathbb{X}$ equipped with two distinct topologies $\mathbb{V}_1$ and $\mathbb{V}_2$. This framework allows for more nuanced analysis of continuity, convergence, and separation properties compared to single-topology spaces. Further extending this structure, Kuratowski (1966) introduced the notion of ideals in topological spaces [2].

An ideal is a nonempty family of subsets of $\mathbb{X}$ satisfying hereditary and finite additive properties, making it a useful tool in generalizing open set operations and defining new types of openness. Let I

be an ideal on $\mathbb{X}$ then $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ is called an ideal bitopological space. If $\mathcal{P}(\mathbb{X})$ is the entire set of subsets of $\mathbb{X}$, a set operator $(.)_i^*: \mathcal{P}(\mathbb{X}) \to \mathcal{P}(\mathbb{X})$ given the title local function of $\mathcal{U}$ with reference to $\mathbb{V}_i$ and I .The clarification of local function is stated as: for $\mathcal{U} \subset \mathbb{X}$, $\mathcal{U}_i^*(\mathbb{V}_i, \mathrm{I})$={x∈ $\mathbb{X}$| $\mathrm{V} \cap \mathcal{U} \notin \mathrm{I}$, for every $\mathrm{V} \in \mathbb{V}_i(x)$} as well as $\mathbb{V}_i(x) = \{\mathrm{V} \in \mathbb{V}_i | x \in \mathrm{V}\}$. For each ideal topological space $(\mathbb{X}, \mathbb{V}, \mathrm{I})$ There a topology $\mathbb{V}^*(\mathrm{I})$ which is more refined than $\mathbb{V}$, generated by the base $\mathcal{B}(\mathrm{I}, \mathbb{V}) = \{\mathrm{V} \setminus \mathrm{I} | \mathrm{V} \in \mathbb{V} \text{ and } \mathrm{I} \in \mathrm{I}\}$, nevertheless generally $\mathcal{B}(\mathrm{I}, \mathbb{V})$ is not constantly a topology [3]. Additionally, note that closure operator for $\mathbb{V}_i^*(\mathrm{I})$ accurate than $\mathbb{V}_i$ is defined as $\mathrm{Cl}_i^*(\mathcal{U}) = \mathcal{U} \cup \mathcal{U}_i^*$. So $\mathrm{int}_i^*(\mathcal{U})$ named the interior of $\mathcal{A}$ in $\mathbb{V}_i^*(\mathrm{I})$, as well $\mathrm{int}_i^*(\mathcal{U}_j^*)$ mentioned as the interior of $\mathcal{U}_j^*$ concerning the topology $\mathbb{V}_i$. The $\mathrm{interior}_{\mathbb{V}_i}$ of $\mathcal{U}$ is indicate by $\mathrm{int}_{\mathbb{V}_i}(\mathcal{U})$.

In 1999 Dontchev [4] constituted the concept of pre-I-open set in ideal topological. In 2011 Rajesh [5], Caldas and Jafari introduced the notation of $\mathrm{pre}_{i,j}\mathrm{I} - \mathrm{open}$ set in ideal bitopological space. In 2011 the notation of pre^*-I-open sets in ideal topological space has been introduced by Ekici and Noiri [6]. And in 2019 Aqeel and Bin-Kuddah [7] formulated the concept of strong pre^*- I-open sets in ideal topological space.

A similar study in 2024 by Bukhatwa and Demiralp also explored the definition and properties of strong $(\mathrm{i}, \mathrm{j}) - \mathrm{semi}^* - \Gamma-$ open sets in ideal bitopological spaces [8].

Preliminaries:

Let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an ideal bitopological space with no separation axioms are assumed in this space, and IBS be the abbreviation for ideal bitopological space, also let OP be the abbreviation for open sets and CS for closed sets. And assume $\mathcal{U}$ is a subset of $\mathbb{X}$, then we denote respectively the interior and the closure of $\mathcal{U}$ by $\mathrm{Int}_i(\mathcal{U})$ and $\mathrm{Cl}_j(\mathcal{U})$ with reference to $\mathbb{V}_i$ for $i = 1,2$.

Definition 1. Assume $\mathcal{U}$ is a subset of the IBS $(\mathbb{X}, \mathbb{V}, \mathrm{I})$, **is called:**

1. pre-I-OS if $\mathcal{U} \subseteq \mathrm{Int}(\mathrm{Cl}^*(\mathcal{U}))$.[9]
2. Pre^*-I-OS if $\mathcal{U} \subseteq \mathrm{Int}^*(\mathrm{Cl}(\mathcal{U}))$.[10]
3. Strong pre^*-I-OS $\mathcal{U} \subseteq \mathrm{Int}^*(\mathrm{Cl}^*(\mathcal{U}))$.[7]

Definition 2. Assume $\mathcal{U}$ is a subset of the IBS $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$, such that $\mathrm{i}, \mathrm{j} = 1,2, \text{and } \mathrm{i} \neq \mathrm{j}$. **is named:**

1. (i, j)I-OS if $\mathcal{U} \subseteq \mathrm{Int}_i(\mathcal{U}_j^*)$. [9]
2. (i, j)-preI-OS if $\mathcal{U} \subseteq \mathrm{Int}_i(\mathrm{Cl}_j^*(\mathcal{U}))$. [5]
3. (i, j)-preI-OC if $\mathrm{Cl}_j^*(\mathrm{Int}_i(\mathcal{U})) \subseteq \mathcal{U}$.[5]
4. (i, j)-αI-OS if $\mathcal{U} \subseteq \mathrm{Int}_i(\mathrm{Cl}_j^*(\mathrm{Int}_i(\mathcal{U})))$. [11]

The following definition are from [9]

Definition 3. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS and $\mathcal{U} \subset \mathbb{X}$, **then we get:**

1. If I={∅}, then $\mathcal{U}_j^*(\mathrm{I}) = \mathrm{Cl}_j(\mathcal{U})$.
2. If I= $\mathcal{P}(\mathbb{X})$, then $\mathcal{U}_j^*(\mathrm{I})$=∅.
3. $\mathcal{U}_j^* \subset \mathrm{Cl}_j(\mathcal{U})$.

Lemma 1. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS and $\mathcal{U} \subset \mathbb{X}$, with reference to any (i,j)-I-OS then we get $\mathcal{U}_j^* = (\mathrm{int}_i(\mathcal{U}_j^*))_j^*$.

Definition 4. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS. Then if $\mathrm{I} \cap \mathbb{V}_i$= {∅}, then I is called codense.

Lemma 2. Assume $\mathcal{U}$ is a subset of the IBS $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$, and $\mathrm{V} \in \mathbb{V}_i$. Then, $\mathrm{V} \cap \mathrm{Cl}_j^*(\mathcal{U}) \subseteq \mathrm{Cl}_j^*(\mathrm{V} \cap \mathcal{U})$.

3 ON $\mathrm{Pre}_{i,j}^*\mathfrak{T}$-OPEN SET

Definition 5. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ is an IBS then a subset $\mathcal{U}$ of $\mathbb{X}$ is named as $Pre_{i,j}^*$I-OS if $\mathcal{U} \subseteq \mathrm{Int}_i^*(\mathrm{Cl}_j(\mathcal{U}))$, in which $\mathrm{i}, \mathrm{j} = 1,2, \text{and } \mathrm{i} \neq \mathrm{j}$. The collection comprised of all $\mathrm{P}_{i,j}^*$I-OS in $\mathbb{X}$ will be denoted by $\mathrm{P}_{ij}^*\mathrm{IO}(\mathbb{X})$.

Example 1. Consider an IBS $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$; $\mathbb{X}$= {a, b, c} and $\mathbb{V}_1$= {∅,{a}, {b}, {a, b},$\mathbb{X}$}, $\mathbb{V}_2$= {∅,{a, c},$\mathbb{X}$}. And if I = {∅, {b}, {c}, {b, c}}. Then {a, b} is a $P_{2,1}^*$I-OS but {b, c} is not a $P_{2,1}^*$I-OS.

Proposition 1. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$is an IBS then a subset $\mathcal{U}$ of $\mathbb{X}$. Then each

1. Every (i,j)I-OS is a $\mathrm{P}_{i,j}^*$I-OS.
2. Every $\mathrm{pre}_{i,j}$I-OS is a $\mathrm{P}_{i,j}^*$I-OS.
3. Every $\alpha_{i,j}$I-OS is a $\mathrm{P}_{i,j}^*$I-OS

Proof.

1. Assume $\mathcal{U}$ is a (i,j)I-OS, so $\mathcal{U} \subseteq \mathrm{Int}_i(\mathcal{U}_j^*) \subseteq \mathrm{Int}_i(Cl_j(\mathcal{U})) \subseteq \mathrm{Int}_i^*(\mathrm{Cl}_j(\mathcal{U}))$.
2. Assume $\mathcal{U}$ is a $\mathrm{pre}_{i,j}$I-OS, so $\mathcal{U} \subseteq \mathrm{Int}_i(\mathrm{Cl}_j^*(\mathcal{U})). \subseteq \mathrm{Int}_i^*(\mathrm{Cl}_j^*(\mathcal{U})) \subseteq \mathrm{Int}_i^*(\mathrm{Cl}_j(\mathcal{U}))$, since $\mathbb{V}_i \subset \mathbb{V}_i{}^*$.
3. The proof follows from definition. □

Corollary 1. Let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS and assume $\mathcal{U}$ is a subset of $\mathbb{X}$, is a $P_{i,j}^*$I-OS, if and only if, there is an open set $\mathcal{V}$ such that $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathrm{Cl}_j(\mathcal{U})$.

Proof. If $\mathcal{U}$ is a $P_{i,j}^{*}$I-OS, so $\mathcal{U} \subseteq \mathrm{Int}_i^*(\mathrm{Cl_j}(\mathcal{U}))$. Let $\mathcal{V} = \mathrm{Int}_i^*\left(\mathrm{Cl_j}(\mathcal{U})\right)$ be (i, j)I-OS. We have $\mathcal{U} \subseteq \mathrm{Int}_i^*(\mathrm{Cl_j}(\mathcal{U})) = \mathcal{V} \subseteq \mathrm{Cl_j}(\mathcal{U})$.

In contrast, if $\mathcal{V}$ is a $\mathrm{P}_{i,j}^*$I-OS such that $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathrm{Cl_j}(\mathcal{U})$, so $\mathrm{Cl_j}(\mathcal{U}) = \mathrm{Cl_j}(\mathcal{V})$. SO, $\mathcal{V} \subseteq \mathrm{Int}_i^*(\mathrm{Cl_j}(\mathcal{V}))$ and hence $\mathcal{U} \subseteq \mathcal{V} \subseteq (\mathrm{Int}_i^*(\mathrm{Cl_j}(\mathcal{V}))) = \left(\mathrm{Int}_i^*(\mathrm{Cl_j}(\mathcal{U})\right)$.Which gives that $\mathcal{U}$ is a $\mathrm{P}_{i,j}^*$I-OS.

Theorem 1. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2,, \mathrm{I})$ is an IBS and $\{U_\alpha : \alpha \in \Lambda\}$a family of subsets of $\mathbb{X}$, which Λ is an arbitrary index set.

1. If $\mathrm{U} \in \mathrm{P}_{ij}^*\mathrm{IO}(\mathbb{X})$ for every $\alpha \in \Lambda$, so $\cup_{\alpha\in\Lambda}\{\mathrm{U}_\alpha : \alpha \in \Lambda\} \in \mathrm{P}_{ij}^*\mathrm{IO}(\mathbb{X})$.
2. If $\mathrm{U} \in \mathrm{P}_{ij}^*\mathrm{IC}(\mathbb{X})$ for every $\alpha \in \Lambda$, then the intercection of any two is $\in \mathrm{P}_{ij}^*\mathrm{IC}(\mathbb{X})$.

Proof. 1-Since $\mathrm{U}_\alpha \in \mathrm{P}_{ij}^*\mathrm{IO}(\mathbb{X})$ for every $\alpha \in \Lambda$, so $\mathrm{U}_\alpha \subseteq \mathrm{Int}_i^*(\mathrm{Cl_j}(\mathrm{U}_\alpha)$ for every $\alpha \in \Lambda$. We get $\cup_{\alpha\in\Lambda}\mathrm{U}_\alpha \subseteq \cup_{\alpha\in\Lambda}(\mathrm{Int}_i^*(\mathrm{Cl_j}(\mathrm{U}_\alpha))) \subseteq \mathrm{Int}_i^*\left(\cup_{\alpha\in\Lambda}\mathrm{Cl_j}(\mathrm{U}_\alpha)\right) \subseteq \mathrm{Int}_i^*\left(\mathrm{Cl_j}(\cup_{\alpha\in\Lambda}\mathrm{U}_\alpha)\right)$.

2-If $\mathcal{U} \in \mathrm{P}_{ij}^*\mathrm{IO}(\mathbb{X})$ and $\mathcal{V} \in \mathbb{V}_i$, we get $\mathcal{U} \subseteq \mathrm{Int}_i^*(\mathrm{Cl_j}(\mathcal{U})$, by Lemma 2 we get $\mathcal{U} \cap \mathcal{V} \subseteq \mathrm{Int}_i^*(\mathrm{Cl_j}(\mathcal{U})) \cap \mathcal{V} = \mathrm{Int}_i^*(\mathrm{Cl_j}(\mathcal{U} \cap \mathcal{V})$. □

A finite intersection of $\mathrm{P}_{ij}^*\mathrm{IO}(\mathbb{X})$ need not to be a $\mathrm{P}_{ij}^*\mathrm{IO}(\mathbb{X})$ in general. As demonstrated by the next example.

Example 2. let $\mathbb{X}$= {a, b, c, d} and $\mathbb{V}_1$= {∅,{c, d},$\mathbb{X}$}, $\mathbb{V}_2$= {∅,{a}, {b}, {a, b},$\mathbb{X}$}. And if I = {∅,{a}, {c}, {a, c}}. Then {a, b, c} and {a, c, d} are $\mathrm{P}_{1,2}^*$I –OS. However, {a, b, c} ∩ {a, c, d} = {a, c}, which is not a $\mathrm{P}_{1,2}^*$I-OS.

Definition 6. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ is an IBS and $\mathcal{U} \subset \mathbb{X}$, so $\mathcal{U}$ is named $\mathrm{pre}_{i,j}^*$I-CS if its complement is $\mathrm{pre}_{i,j}^*$I-OS. All $\mathrm{P}_{i,j}^*$I-CS in $\mathbb{X}$ will denoted by $\mathrm{P}_{ij}^*\mathrm{IC}(\mathbb{X})$.

Proposition 2. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$is an IBS then a subset $\mathcal{U}$ of $\mathbb{X}$. **Then each:**

1. Every $\mathrm{pre}_{i,j}$I-CS is a $\mathrm{P}_{i,j}^*$I-CS.
2. Every $\alpha_{i,j}$I-OS is a $\mathrm{P}_{i,j}^*$I-OS.

Proof. Obvious.

Theorem 2. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ is an IBS and $\mathcal{U} \subset \mathbb{X}$, then $\mathcal{U}$ is a $\mathrm{P}_{i,j}^*$I-CS if and only if, $\mathrm{Cl}_j(\mathrm{Int}_i^*(\mathcal{U})) \subseteq \mathcal{U}$.

Proof. Assume $\mathcal{U}$ is a $\mathrm{P}_{i,j}^*$I-CS of $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$, then $(\mathbb{X} - \mathcal{U})$ is a $P_{i,j}^*$I-OS and therefore $(\mathbb{X} - \mathcal{U}) \subseteq \mathrm{Int}_i^*\left(\mathrm{Cl_j}(\mathbb{X} - \mathcal{U})\right) = \mathbb{X} - \mathrm{Cl}_j(\mathrm{Int}_i^*(\mathcal{U}))$. For that, we get $\mathrm{Cl}_j(\mathrm{Int}_i^*(\mathcal{U})) \subseteq \mathcal{U}$.

In contrast, if $\mathrm{Cl_j}(\mathrm{Int}_i^*(\mathcal{U})) \subseteq \mathcal{U}$, so $(\mathbb{X} - \mathcal{U}) \subseteq \mathrm{Int}_i^*\left(\mathrm{Cl_j}(\mathbb{X} - \mathcal{U})\right)$, we get $(\mathbb{X} - \mathcal{U})$ is a $\mathrm{P}_{i,j}^*$I-OS. Therefore, $\mathcal{U}$ is $\mathrm{P}_{i,j}^*$I-CS. □

Theorem 3. Let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS and assume $\mathcal{U}$ is a subset of $\mathbb{X}$, is a $P_{i,j}^*$I-CS, if and only if, there is an closed set $\mathcal{V}$ such that $\mathrm{Int}_i^*(\mathcal{U}) \subseteq \mathcal{V} \subseteq \mathcal{U}$.

Proof. If $\mathcal{U}$ is a $\mathrm{P}_{i,j}^*$I-CS, so $\mathrm{Cl_j}(\mathrm{Int}_i^*(\mathcal{U})) \subseteq \mathcal{U}$. Let $\mathcal{V} = \mathrm{Cl_j}(\mathrm{Int}_i^*(\mathcal{U}))$ be (i, j)I-CS. We have $\mathrm{Int}_i^*(\mathcal{U}) \subseteq \mathrm{Cl_j}(\mathrm{Int}_i^*(\mathcal{U})) = \mathcal{V} \subseteq \mathcal{U}$.

In contrast, if $\mathcal{V}$ is a $\mathrm{P}_{i,j}^*$I-CS such that $\mathrm{Int}_i^*(\mathcal{U}) \subseteq \mathcal{V} \subseteq \mathcal{U}$, so $\mathrm{Int}_i^*(\mathcal{U}) = \mathrm{Int}_i^*(\mathcal{V})$. So, $\mathrm{Cl_j}(\mathrm{Int}_i^*(\mathcal{V})) \subseteq \mathcal{V}$ and hence $\mathrm{Int}_i^*(\mathcal{U}) \subseteq \mathrm{Cl_j}(\mathrm{Int}_i^*(\mathcal{U})) = \mathrm{Cl_j}(\mathrm{Int}_i^*(\mathcal{V}) \subseteq \mathcal{V} \subseteq \mathcal{U}$. Which gives that $\mathcal{U}$ is a $\mathrm{P}_{i,j}^*$I-CS.

4 On strong $\mathrm{Pre}_{i,j}^*\mathfrak{T}$-open set:

Definition 7. A subset $\mathcal{U}$ of an IBS $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ is named as strong $\mathrm{Pre}_{i,j}^*$I-OS set if $\mathcal{U} \subseteq \mathrm{Int}_i^*(\mathrm{Cl}_j^*(\mathcal{U}))$, in which $\mathrm{i}, \mathrm{j} = 1{,}2$. Such that $\mathrm{i} \neq \mathrm{j}$. The collection comprised of all $\mathrm{SP}_{i,j}^*$I-open sets in $\mathbb{X}$ will dented by $\mathrm{SP}_{ij}^*\mathrm{IO}(\mathbb{X})$.

Example 3. Consider an IBS $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$; $\mathbb{X}$ = {a, b, c} and $\mathbb{V}_1$= {∅,{c},$\mathbb{X}$} , $\mathbb{V}_2$= {∅, {a}, {a, c}, $\mathbb{X}$ }. And if I = {∅, {b}, {c}, {b, c}}. Then {a, c} is a $SP_{2,1}^*$I-OS and {b, c}is not a $SP_{2,1}^*$I-OS.

4.1 Proposition 3. Let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS. Assume $\mathcal{U}$ is a subset of $\mathbb{X}$, then:

1. Every (i,j)I-OS is a $\mathrm{SP}_{i,j}^*$I-OS.
2. Every $\mathrm{pre}_{i,j}$I-OS is a $\mathrm{SP}_{i,j}^*$I-OS.
3. Every $\mathrm{SP}_{i,j}^*$I-OS is a $\mathrm{P}_{i,j}^*$I-OS.
4. Every $\alpha_{i,j}$I-OS is a $\mathrm{SP}_{i,j}^*$I-OS.

Proof. (1) Assume $\mathcal{U}$ is a (i,j)I-OS, so $\mathcal{U} \subseteq \mathrm{Int_i}\left(\mathcal{U}_j^*\right) \subseteq \mathrm{Int}_i\left(\mathcal{U}_j^* \cup \mathcal{U}\right) \subseteq \mathrm{Int}_i(\mathrm{Cl}_j^*(\mathcal{U})) \subseteq \mathrm{Int}_i^*(\mathrm{Cl}_j^*(\mathcal{U}))$.

(2),(3),(4) Comes from Proposition 1 and Definitions 7.

Let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS. Assume $\mathcal{U}$ is a subset of $\mathbb{X}$, **then we get this diagram:**

$$\text{(i,j)-}I - \mathrm{OS} \to \alpha_{i,j}I - \mathrm{OS} \to pre_{i,j}\mathrm{I} - \mathrm{OS} \to SP_{i,j}^*\mathrm{I} - \mathrm{OS} \to P_{i,j}^*\mathrm{I} - \mathrm{OS}$$ □

Generally, the opposites of proposition 7 is not accurate. as demonstrated by the next example.

Example 4. Consider an IBS $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$; $\mathbb{X}$ = {a, b, c, d} and $\mathcal{T}_1$= {∅ ,{c, d}, $\mathbb{X}$} , $\mathcal{T}_2$= {∅,,{a}, {b}, {a, b}, $\mathbb{X}$ }.And if I = {∅,{a}, {c} {a, c}, X}. Then {a, b, c} is a $P_{2,1}^*$I-OS however, it is not a $SP_{2,1}^*$I-OS.

By using this example, we get that {a,d} is a $SP_{2,1}^*$I-OS however, it is not a $pre_{2,1}$I-OS.

Corollary 2. Let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS and assume $\mathcal{U}$ is a subset of $\mathbb{X}$, is a $SP^*_{i,j}$I-OS, if and only if, there is an open set $\mathcal{V}$ such that $\mathcal{U} \subseteq \mathcal{V} \subseteq Cl^*_j(\mathcal{U})$.

Proof. If $\mathcal{U}$ is a $SP^*_{i,j}$I-OS, so $\mathcal{U} \subseteq Int^*_i(Cl^*_j(\mathcal{U})$. Let $\mathcal{V} = Int^*_i(Cl^*_j(\mathcal{U}))$be (i, j)I-OS. We have $\mathcal{U} \subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{U}))= \mathcal{V} \subseteq Cl^*_j(\mathcal{U})$.

In contrast, if $\mathcal{V}$ is a $P^*_{i,j}$I-OS such that $\mathcal{U} \subseteq \mathcal{V} \subseteq Cl^*_j(\mathcal{U})$, so $Cl^*_j(\mathcal{U}) = Cl^*_j(\mathcal{V})$. SO, $\mathcal{V} \subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{V}))$ and hence $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{V}))= \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{U}))$.Which gives that $\mathcal{U}$ is a $SP^*_{i,j}$I-OS.

Theorem 4. Let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS and assume $\mathcal{U}$ and $\mathcal{V}$ are subsets of $\mathbb{X}$. If $\mathcal{U}$ is a $SP^*_{i,j}$I-OS and is a (i, j) - preI-OS, then $\mathcal{U} \cup \mathcal{V}$ is a $SP^*_{i,j}$I-OS.

Proof.

$$\begin{aligned}\mathcal{U} \cup \mathcal{V} &\subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{U})) \cup \mathrm{Int}_i(\mathrm{Cl}^*_j(\mathcal{V})).\\ &\subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{U})) \cup \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{V}))\\ &\subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{U}) \cup \mathrm{Cl}^*_j(\mathcal{V}))\\ &\subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{U} \cup \mathcal{V})).\end{aligned}$$

So. $\mathcal{U} \cup \mathcal{V}$ is a $SP^*_{i,j}$I-OS. □

Theorem 5. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2,, \mathrm{I})$ is an IBS and $\{U_\alpha : \alpha \in \Lambda\}$a family of subsets of $\mathbb{X}$, which Λ is an arbitrary index set.

1. If $\mathrm{U} \in SP^*_{ij}\mathrm{IO}(\mathbb{X})$ for every $\alpha \in \Lambda$, so $\cup_{\alpha\in\Lambda}\{\mathrm{U}_\alpha : \alpha \in \Lambda\} \in \mathrm{SP}^*_{ij}\mathrm{IO}(\mathbb{X})$.
2. If $\mathrm{U}, \mathrm{V} \in \mathrm{SP}^*_{ij}\mathrm{IC}(\mathbb{X})$ for every $\mathrm{V} \in \Lambda$, then $U \cup \mathrm{V} \in \mathrm{SP}^*_{ij}\mathrm{IC}(\mathbb{X})$.

Proof.

1. Since $\mathrm{U}_\alpha \in \mathrm{SP}^*_{ij}\mathrm{IO}(\mathbb{X})$ for every $\alpha \in \Lambda$, so $\mathrm{U}_\alpha \subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathrm{U}_\alpha))$ for every $\alpha \in \Lambda$. We get $\cup_{\alpha\in\Lambda}\mathrm{U}_\alpha \subseteq \cup_{\alpha\in\Lambda}(\mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathrm{U}_\alpha)))$

$$\begin{aligned}&\subseteq \mathrm{Int}^*_i\left(\cup_{\alpha\in\Lambda}\mathrm{Cl}^*_j(\mathrm{U}_\alpha)\right)\\ &\subseteq \mathrm{Int}^*_i(\cup_{\alpha\in\Lambda}(\mathrm{U}_\alpha \cup {\mathrm{U}_\alpha}^*))\\ &\subseteq \mathrm{Int}^*_i(\cup_{\alpha\in\Lambda}\mathrm{U}_\alpha \cup \cup_{\alpha\in\Lambda}{\mathrm{U}_\alpha}^*)\\ &\subseteq \mathrm{Int}^*_i((\cup_{\alpha\in\Lambda}\mathrm{U}_\alpha) \cup (\cup_{\alpha\in\Lambda}\mathrm{U}_\alpha)^*).\\ &\subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j((\cup_{\alpha\in\Lambda}\mathrm{U}_\alpha)).\end{aligned}$$

So, $\cup_{\alpha\in\Lambda}\mathrm{U}_\alpha \in \mathrm{SP}^*_{ij}\mathrm{IO}(\mathbb{X})$.

2. If $\mathcal{U} \in \mathrm{SP}^*_{ij}\mathrm{IO}(\mathbb{X})$ and $\mathcal{V} \in \mathbb{V}_i$, we get $\mathcal{U} \subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{U}))$, by Lemma 2 we get $\mathcal{U} \cap \mathcal{V} \subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{U})) \cap \mathcal{V} \subseteq \mathrm{Int}^*_i(\mathrm{Cl}^*_j(\mathcal{U} \cap \mathcal{V}))$. □

Generally, A finite intersection of a $\mathrm{SP}^*_{ij}\mathrm{IO}(\mathbb{X})$ need not to be a $\mathrm{SP}^*_{ij}\mathrm{IO}(\mathbb{X})$. as demonstrated by the next example.

Definition 8. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ is an IBS and $\mathcal{U} \subset \mathbb{X}$, then $\mathcal{U}$ is named a strong $P^*_{i,j}$I-CS if it is complement is a $SP^*_{i,j}$I-OS. All $SP^*_{i,j}$I-CS in $\mathbb{X}$ will denoted by $\mathrm{SP}^*_{ij}\mathrm{IC}(\mathbb{X})$.

Theorem 6. If $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ is an IBS and $\mathcal{U} \subset \mathbb{X}$, then $\mathcal{U}$ is a $SP^*_{i,j}$I-CS if and only if, $\mathrm{Cl}^*_j(\mathrm{Int}^*_i(\mathcal{U})) \subseteq \mathcal{U}$.

Proof. Assume $\mathcal{U}$ is a $SP^*_{i,j}$I-CS of $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$, then ($\mathbb{X}$ -$\mathcal{U}$) is a $SP^*_{i,j}$I-OS and since $\mathbb{X} - \mathcal{U} \subseteq \mathrm{int}^*_i(\mathrm{Cl}^*_j(\mathbb{X} - \mathcal{U})) = \mathbb{X} - \mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U}))$. Thus, we get $\mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U})) \subseteq \mathcal{U}$.

In contrast, let $\mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U})) \subseteq \mathcal{U}$, then $(\mathbb{X} - \mathcal{U}) \subseteq \mathrm{Cl}^*_j\left(\mathrm{int}^*_i(\mathbb{X} - \mathcal{U})\right)$. We get $(\mathbb{X} - \mathcal{U})$ is a $SP^*_{i,j}$I-OS. Therefore $\mathcal{U}$ is a $SP^*_{i,j}$I-CS. □

Theorem 7. let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be and IBS. If I is condense, then $\mathcal{U}$ is a $SP^*_{i,j}$I-CS, if and only if, $\mathrm{Cl}^*_j(int_i(\mathcal{U})) \subseteq \mathcal{U}$.

Proof. Assume $\mathcal{U}$ is a $SP^*_{i,j}\mathrm{I} - \mathrm{CS}$ of $\mathbb{X}$, then $\mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U})) \subseteq \mathcal{U}$. So $\mathrm{Cl}^*_j(int_i(\mathcal{U})) \subseteq \mathcal{U}$. In contrast, assume $\mathcal{U}$ is any subset of $\mathbb{X}$, such that $\mathrm{Cl}^*_j(int_i(\mathcal{U})) \subseteq \mathcal{U}$. Then $\mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U})) \subseteq \mathcal{U}$, which gives that $\mathcal{U}$ is a $SP^*_{i,j}$I-CS. □

Theorem 8. let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS. $\mathcal{U}, \mathcal{V} \subset \mathbb{X}$, then If $\mathcal{U}$ is a $SP^*_{i,j}$I-CS and $\mathcal{V}$ is a (i,j)-preI-CS, then $\mathcal{U} \cap \mathcal{V}$ is a $SP^*_{i,j}$I-CS.

Proof. Assume $\mathcal{U} \in SP^*_{ij}IO(\mathbb{X})$ and $\mathcal{V}$ is a (i,j)-preI-CS, we get $\mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U})) \subseteq \mathcal{U}$, So, $\mathcal{U} \cap \mathcal{V} \supseteq \mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U})) \cap \mathrm{Cl}^*_j(\mathrm{Int}_i(\mathcal{V}))$

$$\begin{aligned}&\supseteq \mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U})) \cap \mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U}))\\ &\supseteq \mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U}) \cap \mathrm{int}^*_i(\mathcal{V}))\\ &\supseteq \mathrm{Cl}^*_j(\mathrm{int}^*_i(\mathcal{U} \cap \mathcal{V})).\end{aligned}$$

So $\mathcal{U} \cap \mathcal{V}$ is a $SP^*_{i,j}$I-CS.

Theorem 9. let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS. $\mathcal{U} \subset \mathbb{X}$, then $\mathcal{U}$ is a $SP^*_{i,j}$I-CS if $\mathcal{U}$ is both a $P^*_{i,j}$I-OS and a $P^*_{i,j}$I-CS.

Proof. Let $\mathcal{U}$ be a $P^*_{\mathrm{i,j}}$I-OS, SO $\mathcal{U} \subseteq \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$, since $\mathcal{U}$ is a $P^*_{\mathrm{i,j}}$I-CS, then $\mathcal{U} \subseteq \mathrm{Int}^*_{\mathrm{i}}(\mathcal{U}) \subseteq \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$. □

Theorem 10. let $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ be an IBS. $\mathcal{U} \subset \mathbb{X}$, then $\mathcal{U}$ is a $SP^*_{\mathrm{i,j}}$I-CS if $\mathcal{U}$ if and only, if there exists $\mathrm{SP}^*_{\mathrm{i,j}}$I-OS $\mathcal{V}$ such that $\mathrm{int}^*_{\mathrm{i}}(\mathcal{U}) \subset \mathcal{V} \subset \mathcal{U}$.

Proof. Assume $\mathcal{U}$ is a $\mathrm{SP}^*_{\mathrm{i,j}}$I-OS, then $\mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U})) \subseteq \mathcal{U}$. Put $\mathcal{V} = \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{int}^*_{\mathrm{i}}(\mathcal{U})$. So $\mathrm{int}^*_{\mathrm{i}}(\mathcal{U}) \subset \mathcal{V} \subset \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U})) \subseteq \mathcal{U}$.

In contrast, let $\mathcal{V}$ be a $\mathrm{SP}^*_{\mathrm{i,j}}$I-OS such that $\mathrm{int}^*_{\mathrm{i}}(\mathcal{U}) \subset \mathcal{V} \subset \mathcal{U}$, then $\mathrm{int}^*_{\mathrm{i}}(\mathcal{U}) = \mathrm{int}^*_{\mathrm{i}}(\mathcal{V})$.And we have $\mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{V})) \subseteq \mathcal{V}$, so $\mathcal{U} \supseteq \mathcal{V} \supseteq \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{V}))= \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U}))$.

Then $\mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U})) \subseteq \mathcal{U}$. □

5 The strong $\mathrm{Pre}^*_{\mathrm{i,j}}\mathfrak{I}$-interior and strong $\mathrm{Pre}^*_{\mathrm{i,j}}\mathfrak{I}$-closure

Definition 9. Assume $\mathcal{U}$ is a subset of the IBS $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ then the strong $\mathrm{Pre}^*_{\mathrm{i,j}}$I-interior of $\mathcal{U}$ is denoted by $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Int}(\mathcal{U})$ which is defined by the union of all $\mathrm{SP}^*_{\mathrm{i,j}}$I-OS of $\mathbb{X}$ contained in $\mathcal{U}$. $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Int}(\mathcal{U})=\{\cup \mathcal{V}: \mathcal{V} \subset \mathcal{U}, \mathcal{V}$ is a $\mathrm{SP}^*_{\mathrm{i,j}}$I-OS$\}$

Theorem 11. Assume $\mathcal{U}$ is a subset of the IBS $(\mathbb{X}, \mathbb{V}_{\mathbf{1}}, \mathbb{V}_{\mathbf{2}}, \mathrm{I})$, $SP^*_{\mathrm{i,j}}\mathrm{I\text{-}Int}(\mathcal{U}) = \mathcal{U} \cap \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$.

Proof. Let $\mathcal{U} \subset \mathbb{X}$,so $\mathcal{U} \cap \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U})) \subset \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U})) = \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))) = \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}) \cap \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))) \subset \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U} \cap \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))))$.

SO, $\mathcal{U} \cap \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$ is a $SP^*_{\mathrm{i,j}}$I-OS contained in $\mathcal{U}$, then $\mathcal{U} \cap \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U})) \subset SP^*_{\mathrm{i,j}}\mathrm{I\text{-}Int}(\mathcal{U})$.

Since $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Int}(\mathcal{U})$ is $\mathrm{SP}^*_{\mathrm{i,j}}$I-OS, then $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Int}(\mathcal{U}) \subset \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I} - \mathrm{Int}(\mathcal{U})) \subset \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$. So, $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Int}(\mathcal{U}) \subset \mathcal{U} \cap \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$. Thus $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Int}(\mathcal{U}) = \mathcal{U} \cap \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$.

Lemma. Let $(\mathbb{X}, \mathbb{V}_{\mathbf{1}}, \mathbb{V}_{\mathbf{2}}, \mathrm{I})$ be an IBS and assume $\mathcal{U}$ is a subset of $\mathbb{X}$, then $\mathcal{U}$ is a $\mathrm{SP}^*_{\mathrm{i,j}}$I-OS if and only if $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Int}(\mathcal{U}) = \mathcal{U}$.

Proof. Assume $\mathcal{U}$ is a $\mathrm{SP}^*_{\mathrm{i,j}}$I-OS, then $\mathcal{U} \subset \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$. Hence $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Int}(\mathcal{U}) \subset \mathcal{U} \cap \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U})) = \mathcal{U}$.

In contrast, let $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Int}(\mathcal{U}) = \mathcal{U}$, and since $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Int}(\mathcal{U}) \subset \mathcal{U} \cap \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U})) = \mathcal{U}$, we get $\mathcal{U} \subset \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$. So $\mathcal{U}$ is a $\mathrm{SP}^*_{\mathrm{i,j}}$I-OS. □

Definition. Assume $\mathcal{U}$ is a subset of the IBS $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$ then the strong $\mathrm{Pre}^*_{\mathrm{i,j}}$I-closure which is denoted by $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Cl}(\mathcal{U})$ and defined by the intersection of all $\mathrm{SP}^*_{\mathrm{i,j}}$I-CS of $\mathbb{X}$ containing $\mathcal{U}$. $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Cl}(\mathcal{U})=\{\cap \mathcal{V}: \mathcal{U} \subset \mathcal{V}, \mathcal{V}$ is a $\mathrm{SP}^*_{\mathrm{i,j}}$I-CS$\}$.

Theorem. If $(\mathbb{X}, \mathbb{V}_{\mathbf{1}}, \mathbb{V}_{\mathbf{2}}, \mathrm{I})$ is an IBS and $\mathcal{U} \subset \mathbb{X}$, then $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Cl}(\mathcal{U}) = \mathcal{U} \cup \mathrm{Cl}^*_{j}(\mathrm{int}^*_{\mathrm{i}}(\mathcal{U})$.

Proof. Let $\mathcal{U} \subset \mathbb{X}$,So $\mathcal{U} \cup \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U})) \supset \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U}))$

$$\supset \mathrm{Cl}^*_{\mathrm{j}}\left(\mathrm{Int}^*_{i}(\mathcal{U})\right) \cup \mathrm{Cl}^*_{\mathrm{j}}\left(\mathrm{Int}^*_{i}(\mathcal{U})\right)$$
$$\supset \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U})) \cup \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U})))$$
$$\supset \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U}) \cup (\mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U}))))$$
$$\supset \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U})) \cup (\mathrm{Int}^*_{i}(\mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U}))))$$
$$\supset \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U} \cup \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U})))).$$

So $\mathcal{U} \cup \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U}))$ is a $SP^*_{\mathrm{i,j}}$I-CS containing $\mathcal{U}$, then $\mathcal{U} \cup \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U})) \subset \mathrm{SP}^*_{\mathrm{i,j}}$I-CS.

In contrast, since $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Cl}(\mathcal{U})$ is $\mathrm{SP}^*_{\mathrm{i,j}}$I-CS then $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Cl}(\mathcal{U}) \supset \mathrm{Cl}^*_{j}(\mathrm{Int}^*_{i}(\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Cl}(\mathcal{U}) \supset \mathrm{Cl}^*_{\mathrm{j}}\left(\mathrm{Int}^*_{i}(\mathcal{U})\right)$, So $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Cl}(\mathcal{U}) \supset \mathcal{U} \cup \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{i}(\mathcal{U}))$.

Thus, $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Cl}(\mathcal{U}) = \mathcal{U} \cup \mathrm{Cl}^*_{j}\left(\mathrm{Int}^*_{i}(\mathcal{U})\right)$. □

Lemma. Let $(\mathbb{X}, \mathbb{V}_{\mathbf{1}}, \mathbb{V}_{\mathbf{2}}, \mathrm{I})$ be an IBS and assume $\mathcal{U}$ is a subset of $\mathbb{X}$, so $\mathcal{U}$ is a $\mathrm{SP}^*_{\mathrm{i,j}}$I-CS if, and only if, $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}Cl}(\mathcal{U}) = \mathcal{U}$.

Proof. Assume $\mathcal{U}$ is a $\mathrm{SP}^*_{\mathrm{i,j}}$I-CS, then $\mathcal{U} \supset \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{U}))$. Then $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Cl}(\mathcal{U}) = \mathcal{U} \cup \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{U})) = \mathcal{U}$.

In contrast, let $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Cl}(\mathcal{U}) = \mathcal{U}$, and since $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Cl}(\mathcal{U}) = \mathcal{U} \cup \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{U}))$, we get $\mathcal{U} \supset \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{U}))$. This suggests that $\mathcal{U}$ is a $\mathrm{SP}^*_{\mathrm{i,j}}$I-CS. □

Theorem. assume $\mathcal{U}$ is a subset of the IBS $(\mathbb{X}, \mathbb{V}_{\mathbf{1}}, \mathbb{V}_{\mathbf{2}}, \mathrm{I})$, **then the next properties are hold:**

1. Let $\mathcal{U}$ be $\mathrm{SP}^*_{\mathrm{i,j}}$I -OS then $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Cl}(\mathcal{U}) = \mathrm{Int}^*_{i}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$.
2. Let $\mathcal{U}$ be $\mathrm{SP}^*_{\mathrm{i,j}}$I -CS, then $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Int}(\mathcal{U}) = \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{U}))$.

Proof. (1) Assume $\mathcal{U}$ is $\mathrm{SP}^*_{\mathrm{i,j}}$I-OS in $\mathbb{X}$, it gives that $\mathcal{U} \subset \mathrm{Int}_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$. So $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Cl}(\mathcal{U})= \mathcal{U} \cup \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U})) = \mathrm{Int}^*_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{U}))$.

(2) Assume $\mathcal{U}$ is $\mathrm{SP}^*_{\mathrm{i,j}}$I-CS in $\mathbb{X}$, it gives that $\mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{U})) \subset \mathcal{U}$. So $\mathrm{SP}^*_{\mathrm{i,j}}\mathrm{I\text{-}\ Int}(\mathcal{U}) = \mathcal{U} \cap \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{U})) = \mathrm{Cl}^*_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{U}))$. □

Theorem. Assume $\mathcal{U}$ is a subset of the IBS $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathrm{I})$, **then the next properties are hold:**

1. $\mathrm{Int}_i^*(\mathrm{SP}_{i,j}^*\mathrm{I}\text{-}\,\mathrm{Cl}(\mathcal{U})) \subseteq \mathrm{Int}_i^*\left(\mathrm{Cl}_j^*(\mathcal{U})\right)$.
2. $\mathrm{Cl}_j^*\left(\mathrm{Int}_i^*(\mathcal{U})\right) \subseteq \mathrm{Cl}_j^*(\mathrm{SP}_{i,j}^*\mathrm{I}\text{-}\,\mathrm{Int}(\mathcal{U}))$.

Proof.

1. We have, $\mathrm{Int}_i^*(\mathrm{SP}_{i,j}^*\mathrm{I}\text{-}\,\mathrm{Cl}(\mathcal{U})) = \mathrm{Int}_i^*(\mathcal{U} \cup \mathrm{Cl}_i^*(\mathrm{Int}_i^*(\mathcal{U}))) \subseteq \mathrm{Int}_i^*\left(\mathcal{U} \cup \mathrm{Cl}_j^*(\mathcal{U})\right) \subseteq \mathrm{Int}_i^*\left(\mathrm{Cl}_j^*(\mathcal{U})\right)$. This indicates that $\mathrm{Int}_i^*(\mathrm{SP}_{i,j}^*\mathrm{I}\text{-}\,\mathrm{Cl}(\mathcal{U})) \subseteq \mathrm{Int}_i^*\left(\mathrm{Cl}_j^*(\mathcal{U})\right)$.
2. We have, $\mathrm{Cl}_j^*\left(\mathrm{Int}_i^*(\mathcal{U})\right) = \mathrm{Cl}_j^*\left(\mathcal{U} \cap \mathrm{Int}_i^*(\mathcal{U})\right) \subseteq \mathrm{Cl}_j^*\left(\mathcal{U} \cap \left(\mathrm{Int}_i^*(\mathrm{Cl}_j^*(\mathcal{U})\right)\right)$. This suggests that $\mathrm{Cl}_j^*\left(\mathrm{Int}_i^*(\mathcal{U})\right) \subseteq \mathrm{Cl}_j^*(\mathrm{SP}_{i,j}^*\mathrm{I}\text{-}\,\mathrm{Int}(\mathcal{U}))$. □

Conclusion:

In this paper, we have introduced and explored the concept of $\mathrm{Pre}_{i,j}^*\mathrm{I}$-open sets and strong $\mathrm{Pre}_{i,j}^*\mathrm{I}$-open sets within the framework of ideal bitopological spaces. These notions contribute to a deeper understanding of topological structures by offering generalized forms that are weaker yet still rich in properties. We also developed the corresponding notions of strong $Pre_{i,j}^*\mathrm{I}$-interior and closure operators, and examined their fundamental characteristics. These results pave the way for further studies on generalized open sets and separation axioms in bitopological and ideal spaces

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