



## On $Pre_{i,j}^* \mathfrak{T}$ -Open Sets and Strong $Pre_{i,j}^* \mathfrak{T}$ -Open Sets in Ideal Bitopological Space

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### المجموعات المفتوحة القوية $Pre_{i,j}^* \mathfrak{T}$ في الفضاءات التبولوجية الثنائية المثالية

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#### Abstract:

Firstly, in this study, we introduce the concept of  $Pre_{i,j}^* \mathfrak{T}$ -open set in ideal bitopological spaces, which is weaker than the concept of  $Pre_{i,j} \mathfrak{T}$ -open set in ideal bitopological spaces. Subsequently, we present the notion of strong  $Pre_{i,j}^* \mathfrak{T}$ -open set in ideal bitopological spaces and discuss some of their properties. Furthermore, based on this new concept, we define the strong  $Pre_{i,j}^* \mathfrak{T}$ -interior and strong  $Pre_{i,j}^* \mathfrak{T}$ -closure, along with an investigation of their various properties.

**Keywords:** Ideal bitopological space,  $Pre_{i,j}^* \mathfrak{T}$ -open set, strong  $Pre_{i,j}^* \mathfrak{T}$ -open set,  $SP_{i,j}^* \mathfrak{T}$ -Int( $\mathcal{U}$ ),  $SP_{i,j}^* \mathfrak{T}$ -Cl( $\mathcal{U}$ ).

#### الملخص:

في هذه الورقة، قدمنا واستكشفنا مفهوم المجموعات المفتوحة من النوع  $Pre_{i,j}^* \mathfrak{T}$  والمجموعات المفتوحة القوية من النوع  $Pre_{i,j}^* \mathfrak{T}$  ضمن إطار الفضاءات التبولوجية الثنائية المثالية. تُسهّم هذه المفاهيم في فهم أعمق للبنى الطبولوجية من خلال تقديم أشكال معمة أضعف ولكنها غنية بالخصائص. كما طورنا المفاهيم المقابلة لمؤثرات الإغلاق والداخلية القوية من النوع  $Pre_{i,j}^* \mathfrak{T}$  ودرسنا خصائصها الأساسية. تُمهّد هذه النتائج الطريق لمزيد من الدراسات حول المجموعات المفتوحة المعمة وبديهيّات الفصل في الفضاءات الطبولوجية الثنائية والمثالية.  
**الكلمات المفتاحية:** الفضاء التبولوجي الثنائي المثالي، 1 لمجموعة المفتوحة من النوع  $Pre_{i,j}^* \mathfrak{T}$ ، المجموعة المفتوحة القوية من النوع  $Pre_{i,j}^* \mathfrak{T}$ ، الداخلية والغلاقة للنوع  $SP_{i,j}^* \mathfrak{T}$ -Int( $\mathcal{U}$ ),  $SP_{i,j}^* \mathfrak{T}$ -Cl( $\mathcal{U}$ ).

#### Introduction:

The study of topological structures has evolved significantly through the integration of new abstract concepts that enhance classical topology. One such advancement is the bitopological space, first introduced by Kelly (1963) [1], where a set  $X$  equipped with two distinct topologies  $\mathfrak{V}_1$  and  $\mathfrak{V}_2$ . This framework allows for more nuanced analysis of continuity, convergence, and separation properties compared to single-topology spaces. Further extending this structure, Kuratowski (1966) introduced the notion of ideals in topological spaces [2].

An ideal is a nonempty family of subsets of  $\mathbb{X}$  satisfying hereditary and finite additive properties, making it a useful tool in generalizing open set operations and defining new types of openness. Let  $\mathfrak{I}$  be an ideal on  $\mathbb{X}$  then  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  is called an ideal bitopological space. If  $\mathcal{P}(\mathbb{X})$  is the entire set of subsets of  $\mathbb{X}$ , a set operator  $(\cdot)_i^*: \mathcal{P}(\mathbb{X}) \rightarrow \mathcal{P}(\mathbb{X})$  given the title local function of  $\mathcal{U}$  with reference to  $\mathbb{V}_i$  and  $\mathfrak{I}$ . The clarification of local function is stated as: for  $\mathcal{U} \subset \mathbb{X}$ ,  $\mathcal{U}_i^*(\mathbb{V}_i, \mathfrak{I}) = \{x \in \mathbb{X} \mid \forall V \cap \mathcal{U} \notin \mathfrak{I}, \text{ for every } V \in \mathbb{V}_i(x)\}$  as well as  $\mathbb{V}_i(x) = \{V \in \mathbb{V}_i \mid x \in V\}$ . For each ideal topological space  $(\mathbb{X}, \mathbb{V}, \mathfrak{I})$  There a topology  $\mathbb{V}^*(\mathfrak{I})$  which is more refined than  $\mathbb{V}$ , generated by the base  $\mathcal{B}(\mathfrak{I}, \mathbb{V}) = \{V \setminus I \mid V \in \mathbb{V} \text{ and } I \in \mathfrak{I}\}$ , nevertheless generally  $\mathcal{B}(\mathfrak{I}, \mathbb{V})$  is not constantly a topology [3]. Additionally, note that closure operator for  $\mathbb{V}_i^*(\mathfrak{I})$  accurate than  $\mathbb{V}_i$  is defined as  $Cl_i^*(\mathcal{U}) = \mathcal{U} \cup \mathcal{U}_i^*$ . So  $int_i^*(\mathcal{U})$  named the interior of  $\mathcal{A}$  in  $\mathbb{V}_i^*(\mathfrak{I})$ , as well  $int_i^*(\mathcal{U}_j^*)$  mentioned as the interior of  $\mathcal{U}_j^*$  concerning the topology  $\mathbb{V}_i$ . The interior  $\mathbb{V}_i$  of  $\mathcal{U}$  is indicate by  $int_{\mathbb{V}_i}(\mathcal{U})$ .

In 1999 Dontchev [4] constituted the concept of pre- $\mathfrak{I}$ -open set in ideal topological. In 2011 Rajesh [5], Caldas and Jafari introduced the notation of  $pre_{i,j}\mathfrak{I}$  – open set in ideal bitopological space. In 2011 the notation of pre\*- $\mathfrak{I}$ -open sets in ideal topological space has been introduced by Ekici and Noiri [6]. And in 2019 Aqeel and Bin-Kuddah [7] formulated the concept of strong pre\*- $\mathfrak{I}$ -open sets in ideal topological space.

A similar study in 2024 by Bukhatwa and Demiralp also explored the definition and properties of strong  $(i, j)$  – semi\* –  $\Gamma$  – open sets in ideal bitopological spaces [8].

#### Preliminaries:

Let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an ideal bitopological space with no separation axioms are assumed in this space, and IBS be the abbreviation for ideal bitopological space, also let OP be the abbreviation for open sets and CS for closed sets. And assume  $\mathcal{U}$  is a subset of  $\mathbb{X}$ , then we denote respectively the interior and the closure of  $\mathcal{U}$  by  $Int_i(\mathcal{U})$  and  $Cl_j(\mathcal{U})$  with reference to  $\mathbb{V}_i$  for  $i = 1, 2$ .

**Definition 1.** Assume  $\mathcal{U}$  is a subset of the IBS  $(\mathbb{X}, \mathbb{V}, \mathfrak{I})$ , **is called:**

1. pre- $\mathfrak{I}$ -OS if  $\mathcal{U} \subseteq Int(Cl^*(\mathcal{U}))$ . [9]
2. Pre\*- $\mathfrak{I}$ -OS if  $\mathcal{U} \subseteq Int^*(Cl(\mathcal{U}))$ . [10]
3. Strong pre\*- $\mathfrak{I}$ -OS  $\mathcal{U} \subseteq Int^*(Cl^*(\mathcal{U}))$ . [7]

**Definition 2.** Assume  $\mathcal{U}$  is a subset of the IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$ , such that  $i, j = 1, 2$ , and  $i \neq j$ . **is named:**

1.  $(i, j)\mathfrak{I}$ -OS if  $\mathcal{U} \subseteq Int_i(\mathcal{U}_j^*)$ . [9]
2.  $(i, j)$ -pre $\mathfrak{I}$ -OS if  $\mathcal{U} \subseteq Int_i(Cl_j^*(\mathcal{U}))$ . [5]
3.  $(i, j)$ -pre $\mathfrak{I}$ -OC if  $Cl_j^*(Int_i(\mathcal{U})) \subseteq \mathcal{U}$ . [5]
4.  $(i, j)$ - $\alpha\mathfrak{I}$ -OS if  $\mathcal{U} \subseteq Int_i(Cl_j^*(Int_i(\mathcal{U})))$ . [11]

The following definition are from [9]

**Definition 3.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an IBS and  $\mathcal{U} \subset \mathbb{X}$ , **then we get:**

1. If  $\mathfrak{I} = \{\emptyset\}$ , then  $\mathcal{U}_j^*(\mathfrak{I}) = Cl_j(\mathcal{U})$ .
2. If  $\mathfrak{I} = \mathcal{P}(\mathbb{X})$ , then  $\mathcal{U}_j^*(\mathfrak{I}) = \emptyset$ .
3.  $\mathcal{U}_j^* \subset Cl_j(\mathcal{U})$ .

**Lemma 1.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an IBS and  $\mathcal{U} \subset \mathbb{X}$ , with reference to any  $(i, j)\mathfrak{I}$ -OS then we get  $\mathcal{U}_j^* = (int_i(\mathcal{U}_j^*))_j^*$ .

**Definition 4.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an IBS. Then if  $\mathfrak{I} \cap \mathbb{V}_i = \{\emptyset\}$ , then  $\mathfrak{I}$  is called codense.

**Lemma 2.** Assume  $\mathcal{U}$  is a subset of the IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$ , and  $V \in \mathbb{V}_i$ . Then,  $V \cap Cl_j^*(\mathcal{U}) \subseteq Cl_j^*(V \cap \mathcal{U})$ .

### 3 ON Pre\* $\mathfrak{I}$ -OPEN SET

**Definition 5.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  is an IBS then a subset  $\mathcal{U}$  of  $\mathbb{X}$  is named as Pre\* $\mathfrak{I}$ -OS if  $\mathcal{U} \subseteq Int_i^*(Cl_j(\mathcal{U}))$ , in which  $i, j = 1, 2$ , and  $i \neq j$ . The collection comprised of all  $P_{i,j}^*\mathfrak{I}$ -OS in  $\mathbb{X}$  will be denoted by  $P_{i,j}^*\mathfrak{I}O(\mathbb{X})$ .

**Example 1.** Consider an IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$ ;  $\mathbb{X} = \{a, b, c\}$  and  $\mathbb{V}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \mathbb{X}\}$ ,  $\mathbb{V}_2 = \{\emptyset, \{a, c\}, \mathbb{X}\}$ . And if  $\mathfrak{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ . Then  $\{a, b\}$  is a  $P_{2,1}^*\mathfrak{I}$ -OS but  $\{b, c\}$  is not a  $P_{2,1}^*\mathfrak{I}$ -OS.

**Proposition 1.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  is an IBS then a subset  $\mathcal{U}$  of  $\mathbb{X}$ . Then each

1. Every  $(i, j)\mathfrak{I}$ -OS is a  $P_{i,j}^*\mathfrak{I}$ -OS.
2. Every  $pre_{i,j}\mathfrak{I}$ -OS is a  $P_{i,j}^*\mathfrak{I}$ -OS.
3. Every  $\alpha_{i,j}\mathfrak{I}$ -OS is a  $P_{i,j}^*\mathfrak{I}$ -OS

**Proof.**

1. Assume  $\mathcal{U}$  is a  $(i, j)\mathfrak{I}$ -OS, so  $\mathcal{U} \subseteq Int_i(\mathcal{U}_j^*) \subseteq Int_i(Cl_j(\mathcal{U})) \subseteq Int_i^*(Cl_j(\mathcal{U}))$ .
2. Assume  $\mathcal{U}$  is a  $pre_{i,j}\mathfrak{I}$ -OS, so  $\mathcal{U} \subseteq Int_i(Cl_j^*(\mathcal{U})) \subseteq Int_i^*(Cl_j^*(\mathcal{U})) \subseteq Int_i^*(Cl_j(\mathcal{U}))$ , since  $\mathbb{V}_i \subset \mathbb{V}_i^*$ .
3. The proof follows from definition.  $\square$

**Corollary 1.** Let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$  be an IBS and assume  $\mathcal{U}$  is a subset of  $\mathbb{X}$ , is a  $P_{ij}^* \mathfrak{T}$ -OS, if and only if, there is an open set  $\mathcal{V}$  such that  $\mathcal{U} \subseteq \mathcal{V} \subseteq Cl_j(\mathcal{U})$ .

**Proof.** If  $\mathcal{U}$  is a  $P_{ij}^* \mathfrak{T}$ -OS, so  $\mathcal{U} \subseteq Int_i^*(Cl_j(\mathcal{U}))$ . Let  $\mathcal{V} = Int_i^*(Cl_j(\mathcal{U}))$  be  $(i, j) \mathfrak{T}$ -OS. We have  $\mathcal{U} \subseteq Int_i^*(Cl_j(\mathcal{U})) = \mathcal{V} \subseteq Cl_j(\mathcal{U})$ .

In contrast, if  $\mathcal{V}$  is a  $P_{ij}^* \mathfrak{T}$ -OS such that  $\mathcal{U} \subseteq \mathcal{V} \subseteq Cl_j(\mathcal{U})$ , so  $Cl_j(\mathcal{U}) = Cl_j(\mathcal{V})$ . SO,  $\mathcal{V} \subseteq Int_i^*(Cl_j(\mathcal{V}))$  and hence  $\mathcal{U} \subseteq \mathcal{V} \subseteq (Int_i^*(Cl_j(\mathcal{V}))) = (Int_i^*(Cl_j(\mathcal{U})))$ . Which gives that  $\mathcal{U}$  is a  $P_{ij}^* \mathfrak{T}$ -OS.

**Theorem 1.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$  is an IBS and  $\{U_\alpha: \alpha \in \Lambda\}$  a family of subsets of  $\mathbb{X}$ , which  $\Lambda$  is an arbitrary index set.

1. If  $U \in P_{ij}^* \mathfrak{T}O(\mathbb{X})$  for every  $\alpha \in \Lambda$ , so  $U_{\alpha \in \Lambda} \{U_\alpha: \alpha \in \Lambda\} \in P_{ij}^* \mathfrak{T}O(\mathbb{X})$ .

2. If  $U \in P_{ij}^* \mathfrak{T}C(\mathbb{X})$  for every  $\alpha \in \Lambda$ , then the intersection of any two is  $\in P_{ij}^* \mathfrak{T}C(\mathbb{X})$ .

**Proof.** 1-Since  $U_\alpha \in P_{ij}^* \mathfrak{T}O(\mathbb{X})$  for every  $\alpha \in \Lambda$ , so  $U_\alpha \subseteq Int_i^*(Cl_j(U_\alpha))$  for every  $\alpha \in \Lambda$ . We get  $U_{\alpha \in \Lambda} U_\alpha \subseteq U_{\alpha \in \Lambda} (Int_i^*(Cl_j(U_\alpha))) \subseteq Int_i^*(U_{\alpha \in \Lambda} Cl_j(U_\alpha)) \subseteq Int_i^*(Cl_j(U_{\alpha \in \Lambda} U_\alpha))$ .

2-If  $U \in P_{ij}^* \mathfrak{T}O(\mathbb{X})$  and  $\mathcal{V} \in \mathbb{V}_i$ , we get  $\mathcal{U} \subseteq Int_i^*(Cl_j(\mathcal{U}))$ , by Lemma 2 we get  $\mathcal{U} \cap \mathcal{V} \subseteq Int_i^*(Cl_j(\mathcal{U})) \cap \mathcal{V} = Int_i^*(Cl_j(\mathcal{U} \cap \mathcal{V}))$ . □

A finite intersection of  $P_{ij}^* \mathfrak{T}O(\mathbb{X})$  need not to be a  $P_{ij}^* \mathfrak{T}O(\mathbb{X})$  in general. As demonstrated by the next example.

**Example 2.** let  $\mathbb{X} = \{a, b, c, d\}$  and  $\mathbb{V}_1 = \{\emptyset, \{c, d\}, \mathbb{X}\}$ ,  $\mathbb{V}_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \mathbb{X}\}$ . And if  $\mathfrak{T} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ . Then  $\{a, b, c\}$  and  $\{a, c, d\}$  are  $P_{1,2}^* \mathfrak{T}$ -OS. However,  $\{a, b, c\} \cap \{a, c, d\} = \{a, c\}$ , which is not a  $P_{1,2}^* \mathfrak{T}$ -OS.

**Definition 6.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$  is an IBS and  $\mathcal{U} \subset \mathbb{X}$ , so  $\mathcal{U}$  is named  $pre_{ij}^* \mathfrak{T}$ -CS if its complement is  $pre_{ij}^* \mathfrak{T}$ -OS. All  $P_{ij}^* \mathfrak{T}$ -CS in  $\mathbb{X}$  will denoted by  $P_{ij}^* \mathfrak{T}C(\mathbb{X})$ .

**Proposition 2.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$  is an IBS then a subset  $\mathcal{U}$  of  $\mathbb{X}$ . Then each:

1. Every  $pre_{ij}^* \mathfrak{T}$ -CS is a  $P_{ij}^* \mathfrak{T}$ -CS.

2. Every  $P_{ij}^* \mathfrak{T}$ -OS is a  $P_{ij}^* \mathfrak{T}$ -OS.

**Proof.** Obvious.

**Theorem 2.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$  is an IBS and  $\mathcal{U} \subset \mathbb{X}$ , then  $\mathcal{U}$  is a  $P_{ij}^* \mathfrak{T}$ -CS if and only if,  $Cl_j(Int_i^*(\mathcal{U})) \subseteq \mathcal{U}$ .

**Proof.** Assume  $\mathcal{U}$  is a  $P_{ij}^* \mathfrak{T}$ -CS of  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$ , then  $(\mathbb{X} - \mathcal{U})$  is a  $P_{ij}^* \mathfrak{T}$ -OS and therefore  $(\mathbb{X} - \mathcal{U}) \subseteq Int_i^*(Cl_j(\mathbb{X} - \mathcal{U})) = \mathbb{X} - Cl_j(Int_i^*(\mathcal{U}))$ . For that, we get  $Cl_j(Int_i^*(\mathcal{U})) \subseteq \mathcal{U}$ .

In contrast, if  $Cl_j(Int_i^*(\mathcal{U})) \subseteq \mathcal{U}$ , so  $(\mathbb{X} - \mathcal{U}) \subseteq Int_i^*(Cl_j(\mathbb{X} - \mathcal{U}))$ , we get  $(\mathbb{X} - \mathcal{U})$  is a  $P_{ij}^* \mathfrak{T}$ -OS. Therefore,  $\mathcal{U}$  is  $P_{ij}^* \mathfrak{T}$ -CS. □

**Theorem 3.** Let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$  be an IBS and assume  $\mathcal{U}$  is a subset of  $\mathbb{X}$ , is a  $P_{ij}^* \mathfrak{T}$ -CS, if and only if, there is an closed set  $\mathcal{V}$  such that  $Int_i^*(\mathcal{U}) \subseteq \mathcal{V} \subseteq \mathcal{U}$ .

**Proof.** If  $\mathcal{U}$  is a  $P_{ij}^* \mathfrak{T}$ -CS, so  $Cl_j(Int_i^*(\mathcal{U})) \subseteq \mathcal{U}$ . Let  $\mathcal{V} = Cl_j(Int_i^*(\mathcal{U}))$  be  $(i, j) \mathfrak{T}$ -CS. We have  $Int_i^*(\mathcal{U}) \subseteq Cl_j(Int_i^*(\mathcal{U})) = \mathcal{V} \subseteq \mathcal{U}$ .

In contrast, if  $\mathcal{V}$  is a  $P_{ij}^* \mathfrak{T}$ -CS such that  $Int_i^*(\mathcal{U}) \subseteq \mathcal{V} \subseteq \mathcal{U}$ , so  $Int_i^*(\mathcal{U}) = Int_i^*(\mathcal{V})$ . So,  $Cl_j(Int_i^*(\mathcal{V})) \subseteq \mathcal{V}$  and hence  $Int_i^*(\mathcal{U}) \subseteq Cl_j(Int_i^*(\mathcal{U})) = Cl_j(Int_i^*(\mathcal{V})) \subseteq \mathcal{V} \subseteq \mathcal{U}$ . Which gives that  $\mathcal{U}$  is a  $P_{ij}^* \mathfrak{T}$ -CS.

#### 4 On strong $Pre_{ij}^* \mathfrak{T}$ -open set:

**Definition 7.** A subset  $\mathcal{U}$  of an IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$  is named as strong  $Pre_{ij}^* \mathfrak{T}$ -OS set if  $\mathcal{U} \subseteq Int_i^*(Cl_j^*(\mathcal{U}))$ , in which  $i, j = 1, 2$ . Such that  $i \neq j$ . The collection comprised of all  $SP_{ij}^* \mathfrak{T}$ -open sets in  $\mathbb{X}$  will denoted by  $SP_{ij}^* \mathfrak{T}O(\mathbb{X})$ .

**Example 3.** Consider an IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$ ;  $\mathbb{X} = \{a, b, c\}$  and  $\mathbb{V}_1 = \{\emptyset, \{c\}, \mathbb{X}\}$ ,  $\mathbb{V}_2 = \{\emptyset, \{a\}, \{a, c\}, \mathbb{X}\}$ . And if  $\mathfrak{T} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ . Then  $\{a, c\}$  is a  $SP_{2,1}^* \mathfrak{T}$ -OS and  $\{b, c\}$  is not a  $SP_{2,1}^* \mathfrak{T}$ -OS.

**4.1 Proposition 3.** Let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$  be an IBS. Assume  $\mathcal{U}$  is a subset of  $\mathbb{X}$ , then:

1. Every  $(i, j) \mathfrak{T}$ -OS is a  $SP_{ij}^* \mathfrak{T}$ -OS.

2. Every  $pre_{ij}^* \mathfrak{T}$ -OS is a  $SP_{ij}^* \mathfrak{T}$ -OS.

3. Every  $SP_{ij}^* \mathfrak{T}$ -OS is a  $P_{ij}^* \mathfrak{T}$ -OS.

4. Every  $P_{ij}^* \mathfrak{T}$ -OS is a  $SP_{ij}^* \mathfrak{T}$ -OS.

**Proof.** (1) Assume  $\mathcal{U}$  is a  $(i, j) \mathfrak{T}$ -OS, so  $\mathcal{U} \subseteq Int_i^*(\mathcal{U}_j^*) \subseteq Int_i^*(\mathcal{U}_j^* \cup \mathcal{U}) \subseteq Int_i^*(Cl_j^*(\mathcal{U})) \subseteq Int_i^*(Cl_j^*(\mathcal{U}))$ .

(2),(3),(4) Comes from Proposition 1 and Definitions 7.

Let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{T})$  be an IBS. Assume  $\mathcal{U}$  is a subset of  $\mathbb{X}$ , then we get this diagram:

$$(i, j) \mathfrak{T} - OS \rightarrow \alpha_{i, j} \mathfrak{T} - OS \rightarrow pre_{i, j} \mathfrak{T} - OS \rightarrow SP_{ij}^* \mathfrak{T} - OS \rightarrow P_{ij}^* \mathfrak{T} - OS \quad \square$$

Generally, the opposites of proposition 7 is not accurate. as demonstrated by the next example.

**Example 4.** Consider an IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$ ;  $\mathbb{X} = \{a, b, c, d\}$  and  $\mathcal{T}_1 = \{\emptyset, \{c, d\}, \mathbb{X}\}$ ,  $\mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \mathbb{X}\}$ . And if  $\mathfrak{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \mathbb{X}\}$ . Then  $\{a, b, c\}$  is a  $SP_{2,1}^* \mathfrak{I}$ -OS however, it is not a  $SP_{2,1}^* \mathfrak{I}$ -OS.

By using this example, we get that  $\{a, d\}$  is a  $SP_{2,1}^* \mathfrak{I}$ -OS however, it is not a  $pre_{2,1} \mathfrak{I}$ -OS.

**Corollary 2.** Let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an IBS and assume  $\mathcal{U}$  is a subset of  $\mathbb{X}$ , is a  $SP_{i,j}^* \mathfrak{I}$ -OS, if and only if, there is an open set  $\mathcal{V}$  such that  $\mathcal{U} \subseteq \mathcal{V} \subseteq Cl_j^*(\mathcal{U})$ .

**Proof.** If  $\mathcal{U}$  is a  $SP_{i,j}^* \mathfrak{I}$ -OS, so  $\mathcal{U} \subseteq Int_i^*(Cl_j^*(\mathcal{U}))$ . Let  $\mathcal{V} = Int_i^*(Cl_j^*(\mathcal{U}))$  be  $(i, j)$ -OS. We have  $\mathcal{U} \subseteq Int_i^*(Cl_j^*(\mathcal{U})) = \mathcal{V} \subseteq Cl_j^*(\mathcal{U})$ .

In contrast, if  $\mathcal{V}$  is a  $P_{i,j}^* \mathfrak{I}$ -OS such that  $\mathcal{U} \subseteq \mathcal{V} \subseteq Cl_j^*(\mathcal{U})$ , so  $Cl_j^*(\mathcal{U}) = Cl_j^*(\mathcal{V})$ . So,  $\mathcal{V} \subseteq Int_i^*(Cl_j^*(\mathcal{V}))$  and hence  $\mathcal{U} \subseteq \mathcal{V} \subseteq Int_i^*(Cl_j^*(\mathcal{V})) = Int_i^*(Cl_j^*(\mathcal{U}))$ . Which gives that  $\mathcal{U}$  is a  $SP_{i,j}^* \mathfrak{I}$ -OS.

**Theorem 4.** Let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an IBS and assume  $\mathcal{U}$  and  $\mathcal{V}$  are subsets of  $\mathbb{X}$ . If  $\mathcal{U}$  is a  $SP_{i,j}^* \mathfrak{I}$ -OS and is a  $(i, j)$ -pre $\mathfrak{I}$ -OS, then  $\mathcal{U} \cup \mathcal{V}$  is a  $SP_{i,j}^* \mathfrak{I}$ -OS.

**Proof.**

$$\begin{aligned} \mathcal{U} \cup \mathcal{V} &\subseteq Int_i^*(Cl_j^*(\mathcal{U})) \cup Int_i^*(Cl_j^*(\mathcal{V})) \\ &\subseteq Int_i^*(Cl_j^*(\mathcal{U})) \cup Int_i^*(Cl_j^*(\mathcal{V})) \\ &\subseteq Int_i^*(Cl_j^*(\mathcal{U} \cup \mathcal{V})) \\ &\subseteq Int_i^*(Cl_j^*(\mathcal{U} \cup \mathcal{V})). \end{aligned}$$

So,  $\mathcal{U} \cup \mathcal{V}$  is a  $SP_{i,j}^* \mathfrak{I}$ -OS. □

**Theorem 5.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  is an IBS and  $\{U_\alpha : \alpha \in \Lambda\}$  a family of subsets of  $\mathbb{X}$ , which  $\Lambda$  is an arbitrary index set.

1. If  $U \in SP_{ij}^* \mathfrak{I} \mathcal{O}(\mathbb{X})$  for every  $\alpha \in \Lambda$ , so  $U_{\alpha \in \Lambda} \{U_\alpha : \alpha \in \Lambda\} \in SP_{ij}^* \mathfrak{I} \mathcal{O}(\mathbb{X})$ .

2. If  $U, V \in SP_{ij}^* \mathfrak{I} \mathcal{C}(\mathbb{X})$  for every  $V \in \Lambda$ , then  $U \cup V \in SP_{ij}^* \mathfrak{I} \mathcal{C}(\mathbb{X})$ .

**Proof.**

1. Since  $U_\alpha \in SP_{ij}^* \mathfrak{I} \mathcal{O}(\mathbb{X})$  for every  $\alpha \in \Lambda$ , so  $U_\alpha \subseteq Int_i^*(Cl_j^*(U_\alpha))$  for every  $\alpha \in \Lambda$ . We get  $U_{\alpha \in \Lambda} U_\alpha \subseteq U_{\alpha \in \Lambda} (Int_i^*(Cl_j^*(U_\alpha)))$

$$\begin{aligned} &\subseteq Int_i^*(U_{\alpha \in \Lambda} Cl_j^*(U_\alpha)) \\ &\subseteq Int_i^*(U_{\alpha \in \Lambda} (U_\alpha \cup U_\alpha^*)) \\ &\subseteq Int_i^*(U_{\alpha \in \Lambda} U_\alpha \cup U_{\alpha \in \Lambda} U_\alpha^*) \\ &\subseteq Int_i^*((U_{\alpha \in \Lambda} U_\alpha) \cup (U_{\alpha \in \Lambda} U_\alpha^*)) \\ &\subseteq Int_i^*(Cl_j^*((U_{\alpha \in \Lambda} U_\alpha))). \end{aligned}$$

So,  $U_{\alpha \in \Lambda} U_\alpha \in SP_{ij}^* \mathfrak{I} \mathcal{O}(\mathbb{X})$ .

2. If  $U \in SP_{ij}^* \mathfrak{I} \mathcal{O}(\mathbb{X})$  and  $\mathcal{V} \in \mathbb{V}_i$ , we get  $U \subseteq Int_i^*(Cl_j^*(U))$ , by Lemma 2 we get  $U \cap \mathcal{V} \subseteq Int_i^*(Cl_j^*(U)) \cap \mathcal{V} \subseteq Int_i^*(Cl_j^*(U \cap \mathcal{V}))$ . □

Generally, A finite intersection of a  $SP_{ij}^* \mathfrak{I} \mathcal{O}(\mathbb{X})$  need not to be a  $SP_{ij}^* \mathfrak{I} \mathcal{O}(\mathbb{X})$ . as demonstrated by the next example.

**Definition 8.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  is an IBS and  $\mathcal{U} \subset \mathbb{X}$ , then  $\mathcal{U}$  is named a strong  $P_{i,j}^* \mathfrak{I}$ -CS if it is complement is a  $SP_{i,j}^* \mathfrak{I}$ -OS. All  $SP_{i,j}^* \mathfrak{I}$ -CS in  $\mathbb{X}$  will denoted by  $SP_{i,j}^* \mathfrak{I} \mathcal{C}(\mathbb{X})$ .

**Theorem 6.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  is an IBS and  $\mathcal{U} \subset \mathbb{X}$ , then  $\mathcal{U}$  is a  $SP_{i,j}^* \mathfrak{I}$ -CS if and only if,  $Cl_j^*(Int_i^*(\mathcal{U})) \subseteq \mathcal{U}$ .

**Proof.** Assume  $\mathcal{U}$  is a  $SP_{i,j}^* \mathfrak{I}$ -CS of  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$ , then  $(\mathbb{X} - \mathcal{U})$  is a  $SP_{i,j}^* \mathfrak{I}$ -OS and since  $\mathbb{X} - \mathcal{U} \subseteq Int_i^*(Cl_j^*(\mathbb{X} - \mathcal{U})) = \mathbb{X} - Cl_j^*(Int_i^*(\mathcal{U}))$ . Thus, we get  $Cl_j^*(Int_i^*(\mathcal{U})) \subseteq \mathcal{U}$ .

In contrast, let  $Cl_j^*(Int_i^*(\mathcal{U})) \subseteq \mathcal{U}$ , then  $(\mathbb{X} - \mathcal{U}) \subseteq Cl_j^*(Int_i^*(\mathbb{X} - \mathcal{U}))$ . We get  $(\mathbb{X} - \mathcal{U})$  is a  $SP_{i,j}^* \mathfrak{I}$ -OS. Therefore  $\mathcal{U}$  is a  $SP_{i,j}^* \mathfrak{I}$ -CS. □

**Theorem 7.** let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be and IBS. If  $\mathfrak{I}$  is condense, then  $\mathcal{U}$  is a  $SP_{i,j}^* \mathfrak{I}$ -CS, if and only if,  $Cl_j^*(Int_i(\mathcal{U})) \subseteq \mathcal{U}$ .

**Proof.** Assume  $\mathcal{U}$  is a  $SP_{i,j}^* \mathfrak{I}$ -CS of  $\mathbb{X}$ , then  $Cl_j^*(Int_i^*(\mathcal{U})) \subseteq \mathcal{U}$ . So  $Cl_j^*(Int_i(\mathcal{U})) \subseteq \mathcal{U}$ . In contrast, assume  $\mathcal{U}$  is any subset of  $\mathbb{X}$ , such that  $Cl_j^*(Int_i(\mathcal{U})) \subseteq \mathcal{U}$ . Then  $Cl_j^*(Int_i^*(\mathcal{U})) \subseteq \mathcal{U}$ , which gives that  $\mathcal{U}$  is a  $SP_{i,j}^* \mathfrak{I}$ -CS. □

**Theorem 8.** let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an IBS.  $\mathcal{U}, \mathcal{V} \subset \mathbb{X}$ , then If  $\mathcal{U}$  is a  $SP_{i,j}^* \mathfrak{I}$ -CS and  $\mathcal{V}$  is a  $(i, j)$ -pre $\mathfrak{I}$ -CS, then  $\mathcal{U} \cap \mathcal{V}$  is a  $SP_{i,j}^* \mathfrak{I}$ -CS.

**Proof.** Assume  $\mathcal{U} \in SP_{i,j}^* \mathfrak{I} \mathcal{O}(\mathbb{X})$  and  $\mathcal{V}$  is a  $(i, j)$ -pre $\mathfrak{I}$ -CS, we get  $Cl_j^*(Int_i^*(\mathcal{U})) \subseteq \mathcal{U}$ , So,  $\mathcal{U} \cap \mathcal{V} \supseteq Cl_j^*(Int_i^*(\mathcal{U})) \cap Cl_j^*(Int_i(\mathcal{V}))$

$$\begin{aligned} &\supseteq Cl_j^*(Int_i^*(\mathcal{U})) \cap Cl_j^*(Int_i^*(\mathcal{U})) \\ &\supseteq Cl_j^*(Int_i^*(\mathcal{U}) \cap Int_i^*(\mathcal{V})) \\ &\supseteq Cl_j^*(Int_i^*(\mathcal{U} \cap \mathcal{V})). \text{ So } \mathcal{U} \cap \mathcal{V} \text{ is a } SP_{i,j}^* \mathfrak{I}\text{-CS.} \end{aligned}$$

**Theorem 9.** let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an IBS.  $\mathcal{U} \subset \mathbb{X}$ , then  $\mathcal{U}$  is a  $SP_{ij}^* \mathfrak{I}$ -CS if  $\mathcal{U}$  is both a  $P_{ij}^* \mathfrak{I}$ -OS and a  $P_{ij}^* \mathfrak{I}$ -CS.

**Proof.** Let  $\mathcal{U}$  be a  $P_{ij}^* \mathfrak{I}$ -OS, SO  $\mathcal{U} \subseteq \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$ , since  $\mathcal{U}$  is a  $P_{ij}^* \mathfrak{I}$ -CS, then  $\mathcal{U} \subseteq \text{Int}_i^*(\mathcal{U}) \subseteq \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$ .

□

**Theorem 10.** let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an IBS.  $\mathcal{U} \subset \mathbb{X}$ , then  $\mathcal{U}$  is a  $SP_{ij}^* \mathfrak{I}$ -CS if  $\mathcal{U}$  if and only, if there exists  $SP_{ij}^* \mathfrak{I}$ -OS  $\mathcal{V}$  such that  $\text{int}_i^*(\mathcal{U}) \subset \mathcal{V} \subset \mathcal{U}$ .

**Proof.** Assume  $\mathcal{U}$  is a  $SP_{ij}^* \mathfrak{I}$ -OS, then  $\text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})) \subseteq \mathcal{U}$ . Put  $\mathcal{V} = \text{Cl}_j^*(\text{int}_i^*(\mathcal{U}))$ . So  $\text{int}_i^*(\mathcal{U}) \subset \mathcal{V} \subset \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})) \subseteq \mathcal{U}$ .

In contrast, let  $\mathcal{V}$  be a  $SP_{ij}^* \mathfrak{I}$ -OS such that  $\text{int}_i^*(\mathcal{U}) \subset \mathcal{V} \subset \mathcal{U}$ , then  $\text{int}_i^*(\mathcal{U}) = \text{int}_i^*(\mathcal{V})$ . And we have  $\text{Cl}_j^*(\text{Int}_i^*(\mathcal{V})) \subseteq \mathcal{V}$ , so  $\mathcal{U} \supseteq \mathcal{V} \supseteq \text{Cl}_j^*(\text{Int}_i^*(\mathcal{V})) = \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))$ .

Then  $\text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})) \subseteq \mathcal{U}$ . □

### 5 The strong $Pre_{ij}^* \mathfrak{I}$ -interior and strong $Pre_{ij}^* \mathfrak{I}$ -closure

**Definition 9.** Assume  $\mathcal{U}$  is a subset of the IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  then the strong  $Pre_{ij}^* \mathfrak{I}$ -interior of  $\mathcal{U}$  is denoted by  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U})$  which is defined by the union of all  $SP_{ij}^* \mathfrak{I}$ -OS of  $\mathbb{X}$  contained in  $\mathcal{U}$ .  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}) = \{ \cup \mathcal{V} : \mathcal{V} \subset \mathcal{U}, \mathcal{V} \text{ is a } SP_{ij}^* \mathfrak{I}\text{-OS} \}$

**Theorem 11.** Assume  $\mathcal{U}$  is a subset of the IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$ ,

$$SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}) =$$

$$\mathcal{U} \cap \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U})).$$

**Proof.** Let  $\mathcal{U} \subset \mathbb{X}$ , so  $\mathcal{U} \cap \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U})) \subset \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U})) = \text{Int}_i^*(\text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))) = \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}) \cap \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))) \subset \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U} \cap \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U})))$ .

SO,  $\mathcal{U} \cap \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$  is a  $SP_{ij}^* \mathfrak{I}$ -OS contained in  $\mathcal{U}$ , then  $\mathcal{U} \cap \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U})) \subset SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U})$ .

Since  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U})$  is  $SP_{ij}^* \mathfrak{I}$ -OS, then  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}) \subset \text{Int}_i^*(\text{Cl}_j^*(SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}))) \subset \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$ . So,  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}) \subset \mathcal{U} \cap \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$ . Thus  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}) = \mathcal{U} \cap \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$ .

**Lemma.** Let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an IBS and assume  $\mathcal{U}$  is a subset of  $\mathbb{X}$ , then  $\mathcal{U}$  is a  $SP_{ij}^* \mathfrak{I}$ -OS if and only if  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}) = \mathcal{U}$ .

**Proof.** Assume  $\mathcal{U}$  is a  $SP_{ij}^* \mathfrak{I}$ -OS, then  $\mathcal{U} \subset \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$ . Hence  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}) \subset \mathcal{U} \cap \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U})) = \mathcal{U}$ .

In contrast, let  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}) = \mathcal{U}$ , and since  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}) \subset \mathcal{U} \cap \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U})) = \mathcal{U}$ , we get  $\mathcal{U} \subset \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$ . So  $\mathcal{U}$  is a  $SP_{ij}^* \mathfrak{I}$ -OS. □

**Definition.** Assume  $\mathcal{U}$  is a subset of the IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  then the strong  $Pre_{ij}^* \mathfrak{I}$ -closure which is denoted by  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U})$  and defined by the intersection of all  $SP_{ij}^* \mathfrak{I}$ -CS of  $\mathbb{X}$  containing  $\mathcal{U}$ .  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) = \{ \cap \mathcal{V} : \mathcal{U} \subset \mathcal{V}, \mathcal{V} \text{ is a } SP_{ij}^* \mathfrak{I}\text{-CS} \}$ .

**Theorem.** If  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  is an IBS and  $\mathcal{U} \subset \mathbb{X}$ , then  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) = \mathcal{U} \cup \text{Cl}_j^*(\text{int}_i^*(\mathcal{U}))$ .

**Proof.** Let  $\mathcal{U} \subset \mathbb{X}$ , So  $\mathcal{U} \cup \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})) \supset \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))$

$$\begin{aligned} & \supset \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})) \cup \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})) \\ & \supset \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})) \cup \text{Cl}_j^*(\text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))) \\ & \supset \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}) \cup (\text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})))) \\ & \supset \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})) \cup (\text{Int}_i^*(\text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})))) \\ & \supset \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U} \cup \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})))) \end{aligned}$$

So  $\mathcal{U} \cup \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))$  is a  $SP_{ij}^* \mathfrak{I}$ -CS containing  $\mathcal{U}$ , then  $\mathcal{U} \cup \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})) \subset SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U})$ .

In contrast, since  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U})$  is  $SP_{ij}^* \mathfrak{I}$ -CS then  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) \supset \text{Cl}_j^*(\text{Int}_i^*(SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}))) \supset \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))$ , So  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) \supset \mathcal{U} \cup \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))$ .

Thus,  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) = \mathcal{U} \cup \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))$ . □

**Lemma.** Let  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$  be an IBS and assume  $\mathcal{U}$  is a subset of  $\mathbb{X}$ , so  $\mathcal{U}$  is a  $SP_{ij}^* \mathfrak{I}$ -CS if, and only if,  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) = \mathcal{U}$ .

**Proof.** Assume  $\mathcal{U}$  is a  $SP_{ij}^* \mathfrak{I}$ -CS, then  $\mathcal{U} \supset \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))$ . Then  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) = \mathcal{U} \cup \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U})) = \mathcal{U}$ .

In contrast, let  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) = \mathcal{U}$ , and since  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) = \mathcal{U} \cup \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))$ , we get  $\mathcal{U} \supset \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))$ .

This suggests that  $\mathcal{U}$  is a  $SP_{ij}^* \mathfrak{I}$ -CS. □

**Theorem.** assume  $\mathcal{U}$  is a subset of the IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$ , then the next properties are hold:

1. Let  $\mathcal{U}$  be  $SP_{ij}^* \mathfrak{I}$ -OS then  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) = \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$ .
2. Let  $\mathcal{U}$  be  $SP_{ij}^* \mathfrak{I}$ -CS, then  $SP_{ij}^* \mathfrak{I}\text{-Int}(\mathcal{U}) = \text{Cl}_j^*(\text{Int}_i^*(\mathcal{U}))$ .

**Proof.** (1) Assume  $\mathcal{U}$  is  $SP_{ij}^* \mathfrak{I}$ -OS in  $\mathbb{X}$ , it gives that  $\mathcal{U} \subset \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$ . So  $SP_{ij}^* \mathfrak{I}\text{-Cl}(\mathcal{U}) = \mathcal{U} \cup \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U})) = \text{Int}_i^*(\text{Cl}_j^*(\mathcal{U}))$ .

(2) Assume  $\mathcal{U}$  is  $SP_{ij}^*\mathfrak{I}$ -CS in  $\mathbb{X}$ , it gives that  $Cl_j^*(Int_i^*(\mathcal{U})) \subset \mathcal{U}$ . So  $SP_{ij}^*\mathfrak{I}-Int(\mathcal{U}) = \mathcal{U} \cap Cl_j^*(Int_i^*(\mathcal{U})) = Cl_j^*(Int_i^*(\mathcal{U}))$ .  $\square$

**Theorem.** Assume  $\mathcal{U}$  is a subset of the IBS  $(\mathbb{X}, \mathbb{V}_1, \mathbb{V}_2, \mathfrak{I})$ , then the next properties are hold:

1.  $Int_i^*(SP_{ij}^*\mathfrak{I}-Cl(\mathcal{U})) \subseteq Int_i^*(Cl_j^*(\mathcal{U}))$ .
2.  $Cl_j^*(Int_i^*(\mathcal{U})) \subseteq Cl_j^*(SP_{ij}^*\mathfrak{I}-Int(\mathcal{U}))$ .

**Proof.**

1. We have,  $Int_i^*(SP_{ij}^*\mathfrak{I}-Cl(\mathcal{U})) = Int_i^*(\mathcal{U} \cup Cl_j^*(Int_i^*(\mathcal{U}))) \subseteq Int_i^*(\mathcal{U} \cup Cl_j^*(\mathcal{U})) \subseteq Int_i^*(Cl_j^*(\mathcal{U}))$ . This indicates that  $Int_i^*(SP_{ij}^*\mathfrak{I}-Cl(\mathcal{U})) \subseteq Int_i^*(Cl_j^*(\mathcal{U}))$ .
2. We have,  $Cl_j^*(Int_i^*(\mathcal{U})) = Cl_j^*(\mathcal{U} \cap Int_i^*(\mathcal{U})) \subseteq Cl_j^*(\mathcal{U} \cap (Int_i^*(Cl_j^*(\mathcal{U})))$ . This suggests that  $Cl_j^*(Int_i^*(\mathcal{U})) \subseteq Cl_j^*(SP_{ij}^*\mathfrak{I}-Int(\mathcal{U}))$ .  $\square$

**Conclusion:**

In this paper, we have introduced and explored the concept of  $Pre_{ij}^*\mathfrak{I}$ -open sets and strong  $Pre_{ij}^*\mathfrak{I}$ -open sets within the framework of ideal bitopological spaces. These notions contribute to a deeper understanding of topological structures by offering generalized forms that are weaker yet still rich in properties. We also developed the corresponding notions of strong  $Pre_{ij}^*\mathfrak{I}$ -interior and closure operators, and examined their fundamental characteristics. These results pave the way for further studies on generalized open sets and separation axioms in bitopological and ideal spaces

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