The North African Journal of Scientific Publishing (NAJSP)
مجلة شمال إفريقيا للنشر العلمي (NAJSP)
E-ISSN: 2959-4820
Volume 4, Issue 2, 2026
Page No: 16-26
Website: https://najsp.com/index.php/home/index
SJIFactor 2024: 5.49 معامل التأثير العربي (AIF) 2025: 0.69 ISI 2024: 0.696

Oscillation Criteria for a Class of Fourth-Order Nonlinear Neutral Delay Differential Equations

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معايير التذبذب لفئة من المعادلات التفاضلية غير الخطية ذات التأخيرالمحايدة من الرتبة الرابعة

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Received: February 07, 2026 | Accepted: March 22, 2026 | Published: April 03, 2026

Abstract:
The oscillatory characteristics of solutions for a class of nonlinear fourth-order neutral delay differential equations are the primary focus of this work. By extending previously established results, we derive new sufficient conditions that guarantee oscillation. These conditions are developed using advanced analytical tools, including Riccati transformations, comparison principles, and integral inequalities. The neutral nature and high order of the equations considered add complexity, particularly due to the delayed terms appearing in both the dependent variable and its derivatives. Several lemmas and theorems are established to support the main results, an example is provided to illustrate the validity and applicability of the obtained criteria. These results significantly generalize upon known oscillation criteria for fourth-order neutral delay differential equations.

Keywords: Differential equation, Oscillation criteria, Fourth-order differential equations, Neutral delay differential equations.

الملخص
يتخذ هذا البحث من دراسة الخصائص التذبذبية للحلول المنتمية إلى فئة من المعادلات التفاضلية المحايدة ذات التأخير غير الخطية من الرتبة الرابعة محورًا أساسيًا له. وإذ ينطلق من توسيع آفاق نتائج راسخة سلفًا، يستنبط شروطًا كافية جديدة كفيلة بضمان تحقق التذبذب. وقد صيغت هذه الشروط بالاستعانة بأدوات تحليلية متطورة، تشمل تحويلات ريكاتي، ومبادئ المقارنة، والمتباينات التكاملية. وتضفي الطبيعة المحايدة للمعادلات محل النظر، إلى جانب ارتفاع رتبتها، بعدًا من التعقيد على الدراسة، لا سيما في ظل ورود حدود تأخيرية في كل من المتغير التابع ومشتقاته. ولدعم النتائج الرئيسة، تم إثبات مجموعة من النتائج والنظريات، مع الاستشهاد بمثال توضيحي يبرهن على صحة المعايير المستخلصة وجدوى تطبيقها. وتُمثل هذه النتائج في مجملها تعميمًا ملموسًا لمعايير التذبذب المتعارف عليها في سياق المعادلات التفاضلية المحايدة ذات التأخير من الرتبة الرابعة.

الكلمات المفتاحية: معادلة تفاضلية، معايير التذبذب، معادلات تفاضلية من الرتبة الرابعة، المعادلات التفاضلية المحايدة ذات التأخير.

Introduction

Differential equations with delays play a fundamental role in modeling a wide range of natural and engineered systems, in which the rate of change of the state variable depends on both current and historical values of the unknown function. Among these, see (Rihan, 2021), (Ospanov, 2018), (Lynch, 2017), (Triq et al, 2023), neutral delay differential equations (NDDEs) are of particular interest because they incorporate delayed terms in both the dependent variable and its derivatives, making them more complex and rich in dynamical behavior. These equations arise in diverse applications, including engineering, physics, biology, and economics, where processes with aftereffects and feedback mechanisms are prevalent, see (Maan & Barde, 2020), (Barde & Maan, 2019), (Liu et al, 2019).

The oscillatory behavior of solutions to NDDEs has been the subject of intensive research due to its theoretical significance and practical implications, we recall the pioneering works of (Moaaz et al, 2020), (Salah et al, 2023), (Elabbasy et al, 2021), (Bazighifan & Cesarano, 2020), (Dzurina & Kotorova, 2009), (Ladas et al, 1972), (Xing et al, 2011), (Zhang et al, 2014). Oscillation theory provides valuable insights into the qualitative nature of solutions without requiring explicit closed-form solutions, which are often difficult or impossible to obtain for high-order nonlinear systems. Over the past decades, numerous studies have developed criteria for oscillation in various classes of neutral differential equations, significantly enriching the field of qualitative analysis.

This work aims to investigate the oscillation of solutions to the following nonlinear fourth-order neutral delay differential equation.

$$\left(r_3(\ell)\left[\left(r_2(\ell)\left(r_1(\ell)f'(\ell)\right)'\right)'\right]^{\alpha}\right)' + g(\ell)y^{\beta}\left(\rho(\ell)\right) = 0, \tag{1}$$

where:

$$f(\ell) = y(\ell) + h(\ell)y\left(k(\ell)\right),$$

here, $\beta, \alpha \in \mathcal{R}^+$ are such that $\beta = \frac{\varsigma}{m}$ and $\alpha = \frac{s}{n}$, with ς, m, s, n being odd positive integers. The functions satisfy the conditions:

i. $r_i \in C^{4-i}([\ell_0,\infty), \mathcal{R}^+), 1 \leq i \leq 3, i \in \mathbb{Z},$

$$\int_{\ell_0}^{\infty} \frac{1}{r_3^{\frac{1}{\alpha}}(\zeta)} d\zeta = \infty, \ \int_{\ell_0}^{\infty} \frac{1}{r_i(\zeta)} d\zeta = \infty, \ 1 \leq i \leq 2, i \in \mathbb{Z}. \tag{2}$$

ii. $k, \rho \in C([\ell_0,\infty), \mathcal{R}^+), k(\ell) < \ell, \rho(\ell) < \ell, \lim_{\ell\to\infty} k(\ell) = \lim_{\ell\to\infty} \rho(\ell) = \infty.$

iii. $h, g \in C([\ell_0,\infty), \mathcal{R}^+), \ g(\ell) > 0, 0 \leq h(\ell) < h_0 < \infty.$

If any solution y of equation (1) is neither eventually positive nor eventually negative, then it is said to be oscillatory. Furthermore, if all solutions of the differential equation are oscillatory, then the differential equation is called oscillatory (Bazighifan et al, 2020).

If all solutions of the equation oscillate, the equation is termed oscillatory. Otherwise, it is said to be non-oscillatory (Agarwal et al, 2002).

Some previous studies on fourth-order differential equations have addressed equation (1) in the cases where $r_1(\ell), r_2(\ell) = 1$ or $\beta = \alpha = 1$. Among these studies is the work of Waed et al. (Muhsin et al, 2025), where equation (1) was examined for $\beta = \alpha = 1$. Similarly, Bazighifan et al. (Bazighifan et al, 2020) considered equation (1) when $r_1(\ell), r_2(\ell) = 1$. Moreover, (moaaz et al, 2023) when $r_1(\ell), r_2(\ell) = 1$ and $\beta = \alpha$.

Within the present study, Theorem 1 extends Theorem 1 established in study (Muhsin et al, 2025), while Theorem 2 extends Theorem 4 presented in study (Bazighifan et al, 2020).

Building on these findings, the present study aims to broaden the scope of research to include fourth-order neutral delay differential equations by deriving new oscillation criteria that generalize and upon previous results. Using advanced mathematical tools, including Riccati techniques and comparison theorems, this research contributes to a deeper understanding of the oscillatory nature of solutions to high-order neutral systems. The results of this study not only strengthen the theoretical framework of oscillation thonly enhance the theoretical framework of oscillation theory but also pave the way for future research on more generalized forms of delay differential equations.

For our analysis, we shall require the following preliminary results:

Lemma 1. (Bazighifan et al, 2020) Assume that $y \in C^m([\ell_0, \infty), \mathcal{R}^+)$ then

$$\frac{y(\ell)}{\ell^m/m!} \geq \frac{y'(\ell)}{\ell^{m-1}/(m-1)!}$$

where y satisfies $y^{(i)}(\ell) > 0, i = 0,1, \ldots, m \ and \ y^{(m+1)}(\ell) < 0.$

Lemma 2. (Muhsin et al, 2025) Let condition (2) hold and $y(\ell)$ be an eventually positive to (1). Then, $(r_3[(r_2(r_1 f')')']^{\alpha})'(\ell) \le 0$ and either

- $f(\ell) \in S^- \Leftrightarrow f(\ell) > 0, f'(\ell) > 0, (r_1 f')'(\ell) < 0,$
  $r_3(\ell)[(r_2(r_1 f')')']^{\alpha}(\ell) > 0;$
- $f(\ell) \in S^+ \Leftrightarrow f(\ell) > 0, f'(\ell) > 0, (r_1 f')'(\ell) > 0,$
  $r_3(\ell)[(r_2(r_1 f')')']^{\alpha}(\ell) > 0.$

Lemma 3. (Elabbasy et al, 2012) If A is nonnegative, $R, F \ge 0$ and $\gamma > 0$ then

$$RA - FA^{\frac{\gamma+1}{\gamma}} \le \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} R^{\gamma+1} F^{-\gamma}$$

Remark 1. S denotes all positive solutions of equation (1); S^+ denotes all positive solutions of equation (1) with positive limit and S^- denotes all positive solutions of equation (1) with negative limit, such that $S = S^+ \cup S^-$.

Oscillation Criteria

For the purpose of clarity, let

$$T_1(\ell) = \int_{\ell_0}^{\ell} \frac{1}{r_1(v)} dv, \qquad T_2(\ell) = \int_{\ell_0}^{\ell} \frac{1}{r_2(v)} dv, \qquad T_3(\ell) = \int_{\ell_0}^{\ell} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} dv,$$

$$T_{23} = \int_{\ell_1}^{\ell} \frac{1}{r_2(s)} \int_{\ell_1}^{v} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} dv ds, \qquad T_{123} = \int_{\ell_1}^{\ell} \frac{1}{r_1(s)} \int_{\ell_1}^{s} \frac{1}{r_2(\xi)} \int_{\ell_1}^{v} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} dv d\xi ds,$$

$$E_1(\ell) = \frac{(1-h_0)^{\frac{\beta}{\alpha}}}{r_2(\ell)} \left(\int_{\ell}^{\infty} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} \left[\int_{v}^{\infty} g(s) ds \right]^{\frac{1}{\alpha}} dv \right) \left[\int_{\ell_1}^{\rho(\ell)} \frac{1}{r_1(v)} dv \right]^{\frac{\beta}{\alpha}}$$

$$E_2(\ell) = g(\ell)(1-h_0)^{\beta} \left[\int_{\ell_1}^{\ell} \frac{1}{r_1(\xi)} \int_{\ell_1}^{\xi} \frac{1}{r_2(s)} \int_{\ell_1}^{v} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} dv\, ds d\xi \right]^{\beta}$$

Lemma 4. Let us consider that $y \in S^+$. Consequently

$$f'(\ell) \ge \frac{1}{r_1(\ell)} r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1 f')')'(\ell) T_{23}(\ell) \tag{3}$$

Proof. Presume that $y \in S^+$. We reach

$$r_2(\ell)(r_1 f')'(\ell) \ge \int_{\ell_1}^{\ell} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} r_3^{\frac{1}{\alpha}}(v)(r_2(r_1 f')')'(v) dv,$$

based on Lemma 2, we conclude

$$r_2(\ell)(r_1 f')'(\ell) \ge r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1 f')')'(\ell) \int_{\ell_1}^{\ell} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} dv,$$

leads to

$$r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1 f')')'(\ell) \int_{\ell_1}^{\ell} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} dv - r_2(\ell)(r_1 f')'(\ell) \le 0,$$

this suggests that,

$$r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1f')')'(\ell)T_3(\ell)-r_2(\ell)(r_1f')'(\ell)\leq 0. \tag{4}$$

It follows that

$$\left(\frac{r_2(\ell)(r_1f')'(\ell)}{T_3(\ell)}\right)'=\frac{T_3(\ell)r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1f')')'(\ell)-r_2(\ell)(r_1f')'(\ell)}{r_3^{\frac{1}{\alpha}}(\ell)T_3^2(\ell)}$$

$$=\frac{1}{r_3^{\frac{1}{\alpha}}(\ell)T_3^2(\ell)}\left[T_3(\ell)r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1f')')'(\ell)-r_2(\ell)(r_1f')'(\ell)\right],$$

according to (4), it follows that

$$\left(\frac{r_2(\ell)(r_1f')'(\ell)}{T_3(\ell)}\right)'\leq 0.$$

Using this data, we conclude that

$$r_1(\ell)f'(\ell)\geq\int_{\ell_1}^{\ell}T_3(v)\frac{r_2(v)(r_1f')'(v)}{r_2(v)T_3(v)}dv$$

$$\geq\frac{r_2(\ell)(r_1f')'(\ell)}{T_3(\ell)}\int_{\ell_1}^{\ell}T_3(v)\frac{1}{r_2(v)}dv$$

$$=\frac{r_2(\ell)(r_1f')'(\ell)}{T_3(\ell)}\int_{\ell_1}^{\ell}\frac{1}{r_2(\xi)}\int_{\ell_1}^{v}\frac{1}{r_3^{\frac{1}{\alpha}}(v)}dvd\xi=\frac{r_2(\ell)(r_1f')'(\ell)}{T_3(\ell)}T_{23}(\ell),$$

produces

$$f'(\ell)\geq\frac{r_2(\ell)(r_1f')'(\ell)}{r_1(\ell)T_3(\ell)}T_{23}(\ell),$$

according to (4), we get

$$f'(\ell)\geq\frac{1}{r_1(\ell)}r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1f')')'(\ell)T_{23}(\ell).$$

Thus, the proof is concluded.
Theorem 1. Suppose that both first-order delay differential equations

$$q'(\ell)+E_1(\ell)q^{\frac{\beta}{\alpha}}\big(\rho(\ell)\big)=0 \tag{5}$$

and

$$q'(\ell)+E_2(\ell)q^{\frac{\beta}{\alpha}}\big(\rho(\ell)\big)=0 \tag{6}$$

are oscillatory, which implies that equation (1) inherits this property.
Proof. Let us consider that y is ultimately positive and satisfies equation (1), say for $\ell\geq\ell_1$. By applying Lemma 3, we conclude that $f(\ell)\in S^+$ or $f(\ell)\in S^-$. Due to the monotonic nature of $r_1(\ell)f'(\ell)$ that

$$f(\ell) \geq \int_{\ell_1}^{\ell} \frac{1}{r_1(v)} r_1(v) f'(v) dv$$

$$\geq r_1(\ell) f'(\ell) \int_{\ell_1}^{\ell} \frac{1}{r_1(v)} dv \tag{7}$$

$$\Rightarrow \quad f(\ell) \geq r_1(\ell) f'(\ell) T_1(\ell).$$

According to the definition of $f(\ell)$, this suggests

$$y(\ell) = f(\ell) - h(\ell) y\big(k(\ell)\big) \geq f(\ell) - h(\ell) f\big(k(\ell)\big)$$

$$\geq \big(1 - h(\ell)\big) f(\ell) \geq (1 - h_0) f(\ell).$$

Combined with equation (1), implies

$$\Big(r_3(\ell)\big[\big(r_2(\ell)\big(r_1(\ell) f'(\ell)\big)'\big)'\big]^{\alpha}\Big)' = -g(\ell) y^{\beta}\big(\rho(\ell)\big)$$

$$\leq -g(\ell)(1 - h_0)^{\beta} f^{\beta}\big(\rho(\ell)\big)\ . \tag{8}$$

Let us first suppose that $f(\ell) \in S^-$. Upon integration (8) from ℓ to ∞, we obtain

$$r_3(\ell)\big[\big(r_2(\ell)\big(r_1(\ell) f'(\ell)\big)'\big)'\big]^{\alpha} \geq \int_{\ell}^{\infty} g(v)(1 - h_0)^{\beta} f^{\beta}\big(\rho(v)\big) dv.$$

Based on this $f\big(\rho(\ell)\big)$ is increasing, the previous inequality transforms into

$$\big(r_2(\ell)\big(r_1(\ell) f'(\ell)\big)'\big)' \geq f^{\frac{\beta}{\alpha}}\big(\rho(\ell)\big) \frac{(1 - h_0)^{\frac{\beta}{\alpha}}}{r_3^{\frac{1}{\alpha}}(\ell)} \left[\int_{\ell}^{\infty} g(v) dv\right]^{\frac{1}{\alpha}}.$$

Performing another integration, we arrive at

$$-(r_1 f')'(\ell) \geq f^{\frac{\beta}{\alpha}}\big(\rho(\ell)\big) \frac{(1 - h_0)^{\frac{\beta}{\alpha}}}{r_2(\ell)} \int_{\ell}^{\infty} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} \left[\int_{v}^{\infty} g(s) ds\right]^{\frac{1}{\alpha}} dv.$$

Merging this inequality with (7), yields

$$-(r_1 f')'(\ell) \geq \big[f'\big(\rho(\ell)\big)\big]^{\frac{\beta}{\alpha}} \frac{r_1^{\frac{\beta}{\alpha}}\big(\rho(\ell)\big)(1 - h_0)^{\frac{\beta}{\alpha}}}{r_2(\ell)} \int_{\ell}^{\infty} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} \left[\int_{v}^{\infty} g(s) ds\right]^{\frac{1}{\alpha}} dv \left[\int_{\ell_1}^{\rho(\ell)} \frac{1}{r_1(v)} dv\right]^{\frac{\beta}{\alpha}}$$

$$= \big[f'\big(\rho(\ell)\big)\big]^{\frac{\beta}{\alpha}} r_1^{\frac{\beta}{\alpha}}\big(\rho(\ell)\big) E_1(\ell)$$

Thus, the function $q(\ell) = r_1(\ell) f'(\ell)$ represents a solution that is eventually positive to the first-order delay differential inequality

$$q'(\ell) + E_1(\ell) q^{\frac{\beta}{\alpha}}\big(\rho(\ell)\big) \leq 0.$$

Consequently, we deduce that related delay equation (5) as well has a positive solution using Philos' theorem (Philos, 1981); this stands in contradiction to the initial assumptions of the theorem. Consider now the case where $f(\ell) \in S^+$. Given that $r_3(\ell)[(r_2(r_1 f')')']^{\alpha}(\ell)$ is decreasing, This yields

$$r_2(\ell)(r_1f')'(\ell) \geq \int_{\ell_1}^{\ell} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} r_3^{\frac{1}{\alpha}}(v)\big(r_2(v)(r_1f')'(v)\big)'dv$$

$$\geq r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1f')')'(\ell)\int_{\ell_1}^{\ell}\frac{1}{r_3^{\frac{1}{\alpha}}(v)}dv$$

$$\Rightarrow r_2(\ell)(r_1f')'^{(\ell)} \geq r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1f')')'(\ell)T_3(\ell).$$

Hence, integrating the last inequality from ℓ_1 to ℓ, yields

$$f'(\ell) \geq \frac{1}{r_1(\ell)}\int_{\ell_1}^{\ell}\frac{1}{r_2(s)}\int_{\ell_1}^{v}\frac{1}{r_3^{\frac{1}{\alpha}}(v)} r_3^{\frac{1}{\alpha}}(v)\big(r_2(v)(r_1f')'(v)\big)'dv\,ds$$

$$f'(\ell) \geq r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1f')')'(\ell)\frac{1}{r_1(\ell)}\int_{\ell_1}^{\ell}\frac{1}{r_2(s)}\int_{\ell_1}^{v}\frac{1}{r_3^{\frac{1}{\alpha}}(v)}dv\,ds.$$

Performing another integration, yields

$$f(\ell) \geq r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1f')')'(\ell)\int_{\ell_1}^{\ell}\frac{1}{r_1(\xi)}\int_{\ell_1}^{\xi}\frac{1}{r_2(s)}\int_{\ell_1}^{v}\frac{1}{r_3^{\frac{1}{\alpha}}(v)}dv\,dsd\xi.$$

We observe that $q^{\frac{1}{\alpha}}(\ell) = r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1f')')'(\ell)$ fulfills

$$f(\ell) \geq q^{\frac{1}{\alpha}}(\ell)\int_{\ell_1}^{\ell}\frac{1}{r_1(\xi)}\int_{\ell_1}^{\xi}\frac{1}{r_2(s)}\int_{\ell_1}^{v}\frac{1}{r_3^{\frac{1}{\alpha}}(v)}dv\,dsd\xi.$$

Substituting the previous estimate into

$$0 \geq (r_3[(r_2(r_1f')')']^{\alpha})'(\ell) + g(\ell)(1-h_0)^{\beta}f^{\beta}\big(\rho(\ell)\big).$$

$$0 \geq (r_3[(r_2(r_1f')')']^{\alpha})'(\ell) +$$

$$g(\ell)(1-h_0)^{\beta}q^{\frac{\beta}{\alpha}}\big(\rho(\ell)\big)\left[\int_{\ell_1}^{\ell}\frac{1}{r_1(\xi)}\int_{\ell_1}^{\xi}\frac{1}{r_2(s)}\int_{\ell_1}^{v}\frac{1}{r_3^{\frac{1}{\alpha}}(v)}dv\,dsd\xi\right]^{\beta}.$$

Observe that $q(\ell)$ represents an eventually positive solution to the first-order retarded differential inequality

$$q'(\ell) + E_2(\ell)q^{\frac{\beta}{\alpha}}\rho(\ell) \leq 0.$$

Consequently, we deduce that related delay equation (6) as well has a positive solution using Philos' theorem (Philos, 1981); this stands in contradiction to the initial assumptions of the theorem. Thus, the proof is concluded.

Corollary 1. Under the assumption that condition (2) holds. If

$$\lim_{\ell\to\infty} inf \int_{k(\ell)}^{\ell} E_j(s)ds > \frac{1}{e}, \tag{9}$$

for $j = 1,2,$ then equation (1) exhibits oscillatory behavior.

Theorem 2. Let $h_0 < 1$ and suppose that $\rho(\ell) \leq \ell$. Provided there exists a positive functions $\omega, \delta \in C^1([x_0, \infty))$ satisfying

$$\int_{\ell_0}^{\infty} \left(\psi(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r_1^{\alpha}(s)(\omega\prime(s))^{\alpha+1}}{T_{23}^{\alpha}(s)\omega^{\alpha}(s)} \right) ds = \infty. \tag{10}$$

And

$$\int_{\ell_0}^{\infty} \left(\tau(s) - \frac{r_1(s)(\delta\prime(s))^2}{4\delta(s)} \right) ds = \infty, \tag{11}$$

for $M_1, M_2 > 0$, where

$$\psi(\ell) = M_1^{\beta-\alpha} \omega(\ell) g(\ell)(1-h_0)^{\beta} \left(\frac{\rho(\ell)}{\ell} \right)^{3\beta},$$

and

$$\tau(\ell) = \frac{(1-h_0)^{\frac{\beta}{\alpha}}}{r_2(\ell)} \delta(\ell) M_2^{\frac{\beta}{\alpha}-1} \int_{\ell}^{\infty} \frac{1}{r_3^{\frac{1}{\alpha}}(z)} \left(\int_{z}^{\infty} g(s) \frac{\rho^{\beta}(s)}{s^{\beta}} ds \right)^{\frac{1}{\alpha}} dz,$$

then, equation (1) is oscillatory.

Proof. Assume that (1) admits nonoscillatory solution in $[\ell_0, \infty)$. Without of generality, we may assume $y \in S^+$. Then , there exists a $\ell_1 \geq \ell_0$, such that $y(\ell) > 0, y(k(\ell)) > 0$ and $y(\rho(\ell)) > 0$ for $\ell \geq \ell_1$. According to the definition of $f(\ell)$, we deduce.

$$y(\ell) = f(\ell) - h(\ell) y(k(\ell)) \geq f(\ell) - h(\ell) f(k(\ell))$$

$$\geq (1 - h(\ell)) f(\ell) \geq (1 - h_0) f(\ell).$$

Taken together with (1), this leads to

$$0 \geq \left(r_3(\ell) \left[\left(r_2(\ell) (r_1(\ell) f'(\ell))' \right)' \right]^{\alpha} \right)' + g(\ell)(1-h_0)^{\beta} f^{\beta}(\rho(\ell)). \tag{12}$$

Given that $(r_1(\ell) f'(\ell))' > 0$, let us define

$$\eta(\ell) = \omega(\ell) \frac{r_3(\ell) \left[\left(r_2(\ell) (r_1(\ell) f'(\ell))' \right)' \right]^{\alpha}}{f^{\alpha}(\ell)} > 0.$$

Differentiating and applying (12), we get

$$\eta'(\ell) \leq \frac{\omega'(\ell)}{\omega(\ell)} \eta(\ell) - \omega(\ell) g(\ell)(1-h_0)^{\beta} \frac{f^{\beta}(\rho(\ell))}{f^{\alpha}(\ell)} -$$

$$\alpha \omega(\ell) \frac{r_3(\ell) \left[(r_2(r_1 f')\prime)\prime(\ell) \right]^{\alpha}}{f^{\alpha+1}(\ell)} f'(\ell). \tag{13}$$

Lemma 1, yields $f(\ell) \geq \frac{\ell}{3} f'(\rho(\ell))$ and hence,

$$\frac{f(\rho(\ell))}{f(\ell)} \geq \frac{\rho^3(\ell)}{\ell^3}. \tag{14}$$

Lemma 4, provides

$$f'(\ell) \geq r_3^{\frac{1}{\alpha}}(\ell)(r_2(r_1 f')')'(\ell)\frac{1}{r_1(\ell)}T_{23}(\ell), \tag{15}$$

by (13), (14) and (15), we find

$$\eta'(\ell) \leq \frac{\omega'(\ell)}{\omega(\ell)}\eta(\ell) - \omega(\ell)g(\ell)(1-h_0)^{\beta} f^{\beta-\alpha}(\ell)\left(\frac{\rho(\ell)}{\ell}\right)^{3\beta} -$$

$$\alpha\frac{T_{23}(\ell)}{r_1(\ell)\omega^{\frac{1}{\alpha}}(\ell)}\eta^{\frac{1+\alpha}{\alpha}}(\ell).$$

In view of the fact that $f'(\ell) > 0$, there exist a $\ell_2 \geq \ell_1$ such that

$$f(\ell) > M, \tag{16}$$

for all $\ell \geq \ell_2$ and a constant $M > 0$. Applying Lemma 3 we obtain

$$\eta'(\ell) \leq -\psi(\ell) + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r_1^{\alpha}(\ell)\big(\omega'(\ell)\big)^{\alpha+1}}{T_{23}^{\alpha}(\ell)\omega^{\alpha}(\ell)}.$$

Therefore

$$\eta(\ell_1) \geq \int_{\ell_1}^{\ell}\left(\psi(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r_1^{\alpha}(s)\big(\omega'(s)\big)^{\alpha+1}}{T_{23}^{\alpha}(s)\omega^{\alpha}(s)}\right)ds,$$

this contradicts equation (10).

Consider now the case where $y \in S^{-}$, integrating (12) from ℓ to z, we find

$$r_3(z)[(r_2(r_1 f')')'(z)]^{\alpha} - r_3(\ell)[(r_2(r_1 f')')'(\ell)]^{\alpha} \leq$$

$$-\int_{\ell}^{z} g(s)(1-h_0)^{\beta} f^{\beta}\big(\rho(s)\big)ds. \tag{17}$$

Lemma 1, implies that $f(\ell) \geq \ell\, f'(\ell)$, thus,

$$f\big(\rho(\ell)\big) \geq \frac{\rho(\ell)}{\ell}f(\ell). \tag{18}$$

For (17), letting $z \to \infty$ and applying (18), we obtain

$$r_2(r_1 f')'(\ell) \leq -(1-h_0)^{\frac{\beta}{\alpha}} f^{\frac{\beta}{\alpha}}(\ell)\int_{\ell}^{\infty}\frac{1}{r_3^{\frac{1}{\alpha}}(z)}\left(\int_{z}^{\infty} g(s)\frac{\rho^{\beta}(s)}{s^{\beta}}ds\right)^{\frac{1}{\alpha}}dz. \tag{19}$$

Now, We introduce

$$\vartheta(\ell) = \delta(\ell)\frac{f'(\ell)}{f(\ell)}.$$

Then $\vartheta(\ell) > 0$ for $\ell \geq \ell_1$. By using (16) and (19), we get

$$\vartheta'(\ell) = \frac{\delta'(\ell)}{\delta(\ell)}\left(\delta(\ell)\frac{r_1(\ell)f'(\ell)}{f(\ell)}\right) + \delta(\ell)\frac{(r_1 f')'(\ell)}{f(\ell)} - \delta(\ell)r_1(\ell)\left(\frac{f'(\ell)}{f(\ell)}\right)^2$$

$$\leq \frac{\delta'(\ell)}{\delta(\ell)}\vartheta(\ell) - \frac{1}{\delta(\ell)r_1(\ell)}\vartheta^2(\ell)$$

$$-\frac{(1-h_0)^{\frac{\beta}{\alpha}}\delta(\ell)}{r_2(\ell)} f^{\frac{\beta}{\alpha}-1}(\ell)\int_{\ell}^{\infty}\frac{1}{r_3^{\frac{1}{\alpha}}(z)}\left(\int_z^{\infty} g(s)\frac{\rho^{\beta}(s)}{s^{\beta}}ds\right)^{\frac{1}{\alpha}} dz.$$

Thus, we obtain

$$\vartheta'(\ell) \leq -\tau(\ell) + \frac{\delta'(\ell)}{\delta(\ell)}\vartheta(\ell) - \frac{1}{\delta(\ell)r_1(\ell)}\vartheta^2(\ell),$$

Applying Lemma 3, we find

$$\vartheta'(\ell) \leq -\tau(\ell) + \frac{r_1(\ell)\left(\delta'(\ell)\right)^2}{4\delta(\ell)}.$$

Then, we obtain

$$\int_{\ell_1}^{\ell}\left(\tau(s) - \frac{r_1(s)\left(\delta'(s)\right)^2}{4\delta(s)}\right)ds \leq \vartheta(\ell_1),$$

this contradicts equation (11). Thus, the proof is concluded.

Example 1. Let us Consider

$$\left(\tfrac{1}{\ell}f'(\ell)\right)''' + \tfrac{4g_0}{\ell^5}y(\rho_0\ell) = 0, \tag{20}$$

Where $f(\ell) = y(\ell) + h_0 y(k_0\ell)$, $r_1(\ell) = \frac{1}{\ell}, r_2(\ell) = r_3(\ell) = 1,\ \beta = \alpha = 1,\ \rho_0, k_0 \in (0,1), g_0 > 0, h_0 \in [0,1)$ and $\ell > 0$.

It is straightforward to verify that for our equation

$$E_1(\ell) = \frac{(1-h_0)^{\frac{\beta}{\alpha}}}{r_2(\ell)}\left(\int_{\ell}^{\infty}\frac{1}{r_3^{\frac{1}{\alpha}}(v)}\left[\int_v^{\infty} g(s)ds\right]^{\frac{1}{\alpha}} dv\right)\left[\int_{\ell_1}^{\rho(\ell)}\frac{1}{r_1(v)}dv\right]^{\frac{\beta}{\alpha}}$$

$$= (1-h_0)\left(\int_{\ell}^{\infty}\int_v^{\infty}\frac{4g_0}{s^5}ds\, dv\right)\int_0^{\rho_0\ell} v dv$$

$$= g_0(1-h_0)\left(\int_{\ell}^{\infty}\frac{4}{4v^4}dv\right)\frac{\rho_0^2}{2}\ell^2$$

$$= g_0(1-h_0)\left(\frac{4}{12\ell^3}\right)\frac{\rho_0^2}{2}\ell^2$$

$$= \frac{g_0\rho_0^2(1-h_0)}{6}\frac{1}{\ell}.$$

Therefore, condition (9) when $j = 1$ reduces to

$$\liminf_{\ell\to\infty}\int_{k(\ell)}^{\ell} E_1(s)ds$$

$$= \lim_{\ell \to \infty} \inf \int_{k_0 \ell}^{\ell} \frac{g_0 \rho_0^2 (1-h_0)}{6} \frac{1}{s} ds$$

$$= \frac{g_0 \rho_0^2 (1-h_0)}{6} \ln \frac{1}{k_0},$$

this holds true when

$$\frac{g_0 \rho_0^2 (1-h_0)}{6} \ln \frac{1}{k_0} > \frac{1}{e}. \tag{21}$$

Analogously, we find

$$E_2(\ell) = g(\ell)(1-h_0)^{\beta} \left[\int_{\ell_1}^{\ell} \frac{1}{r_1(\xi)} \int_{\ell_1}^{\xi} \frac{1}{r_2(s)} \int_{\ell_1}^{v} \frac{1}{r_3^{\frac{1}{\alpha}}(v)} dv\, ds d\xi \right]^{\beta}$$

$$= \frac{4g_0}{\ell^5}(1-h_0) \int_{\ell_1}^{\ell} \xi \int_{\ell_1}^{\xi} \int_{\ell_1}^{v} dv\, ds d\xi$$

$$= \frac{4g_0}{\ell^5}(1-h_0) \frac{1}{8} \ell^4$$

$$= \frac{g_0 (1-h_0)}{2} \frac{1}{\ell}.$$

Therefore, condition (9) when $j = 2$ reduces to

$$\lim_{\ell \to \infty} \inf \int_{k(\ell)}^{\ell} E_2(s) ds$$

$$\lim_{\ell \to \infty} \inf \int_{k_0 \ell}^{\ell} \frac{g_0 (1-h_0)}{2} \frac{1}{s} ds$$

$$\frac{g_0 (1-h_0)}{2} \ln \frac{1}{k_0},$$

which is satisfies when

$$\frac{g_0 (1-h_0)}{2} \ln \frac{1}{k_0} > \frac{1}{e}. \tag{22}$$

By virtue, Corollary 1, equation (20) is oscillatory if conditions (21) and (22) are fulfilled.

Conclusion

In this study, we have established new oscillation criteria for a class of fourth-order nonlinear neutral delay differential equations. By employing advanced analytical techniques such as Riccati transformation methods, integral inequalities, and comparison theorems, we have derived sufficient conditions under which all solutions of the considered equation are oscillatory. Our results extend and generalize several known oscillation results in the literature by incorporating more general nonlinearities, delay functions, and coefficient conditions. Specifically, the inclusion of delayed terms in both the dependent variable and its derivatives introduces a level of complexity that has been effectively addressed through rigorous mathematical analysis. An illustrative example is provided to highlight the significance and applicability of the main findings. The derived criteria are not only theoretically significant but also offer potential for application in various scientific and engineering models where delay effects and neutral dynamics are present. Future research may consider the extension of these results to equations with distributed delays, impulsive effects, or equations defined on time scales.

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