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New Upper and Lower Bounds for Bateman's G-function

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حدود عليا وسفلى جديدة لدالة جي- بتمان

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Abstract:
This paper presents two new rational approximations for Bateman's G-function, derived using Pad´e approximation techniques. The proposed functions, $f_{18}(x) = \frac{x^{-2}}{2(1+\sum_{i=1}^{9} b_i x^{-2i})}$ *and* $f_{20}(x) = \frac{x^{-2}}{2(1+\sum_{i=1}^{10} b_i x^{-2i})}$, are rigorously proven to serve as sharp lower and upper bounds for $G(x) - x^{-1}$ over the domains $x > 1$ and $x > \frac{3}{2}$, respectively, where $b_1 = \frac{1}{2}$, and

$$b_i = -2\left(\frac{(2^{2i+2}-1)B_{2i+2}}{i+1} + \sum_{s=1}^{i-1} \frac{(2^{2i-2s+2})B_{2i+2s+2}}{i-s+1} b_s\right), \qquad i > 1.$$

Through careful mathematical analysis, we demonstrate that these bounds converge uniformly to $G(x) - x^{-1}$ and exhibit favorable analytic properties. Our results significantly improve the approximation framework for Bateman's G –function, enhancing its applicability in theoretical physics and special function theory. Future studies should explore extensions to complex domains and higher-order approximations.

Keywords: The Pad´e approximation, Bateman's G –function, digamma function, bounds.

الملخص:
هذه الدراسة تقدم حدودا جديدة لدالة جي- بتمان Bateman's G-Function باستخدام صيغة بادي Padé
$G(x) \sim \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(2^{2n-}-1)B_{2n}}{nx^{2n}}, \quad x \to \infty$ تساهم هذه الدراسة في تسليط الضوء على سلوك دالتين من عائلة الدوال $f(x)_{2n} = \frac{x^{-2}}{2(1+\sum_{i=1}^{n} b_j x^{-2i})}$ و هذه النتائج يمكن توظيفها في في دراسة مفاهيم التقارب النقطي او المنتظم لمتابعة من الدوال المتقاربة لدالة جي- بتمان.

الكلمات المفتاحية: تقريب باديه، دالة باتمان G، دالة ديغاما، الحدود.

Introduction:

Bateman's G –function is a special function introduced by the mathematician Harry Bateman [1, 2] and it is used in the context of mathematical physics, applied mathematics, and engineering. It appears in heat conduction, wave propagation, and other physical scenarios where differential equations are solved using integral transforms. The general definitions are as follows:

$$G(x) = 2\int_0^{\infty} \frac{e^{-xv}}{1+e^{-v}}dv, \qquad x > 0 \tag{1}$$

And:

$$G(x) = 2\int_0^{1} \frac{t^{x-1}}{1+t}dt, \qquad x > 0 \tag{2}$$

Bateman's G-function is a generalization and can often be used to represent other special functions like the diamma function and is defined by:

$$G(x) = \psi\left(\frac{1+x}{2}\right) - \psi\left(\frac{x}{2}\right) \qquad x \in R - \{0,-1,-2 \ldots\}. \tag{3}$$

The function $G(x)$ satisfies:

$$G(x) + G(x+1) = \frac{2}{x}. \tag{4}$$

In [3], Mahmoud and Agarwal presented the asymptotic formula:

$$G(x) \sim \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(2^{2n-}-1)B_{2n}}{nx^{2n}}, \qquad x \to \infty \tag{5}$$

where the B_n $, n = 0, 1, 2, \ldots$ are the Bernoulli numbers.

Recently in 2022 [4], Mahmoud, Talat, and Ahfaf introduced the following approximation family:

$$f(x)_{2n} = \frac{x^{-2}}{2(1+\sum_{i=1}^{n} b_j x^{-2i})}, \tag{6}$$

where $q1 = \frac{1}{2}$, and:

$$q_j = -2\left(\frac{(2^{2j+2}-1)B_{2j+2}}{j+1} + \sum_{v-1}^{j-1} \frac{(2^{2j-2v+2}-1)B_{2j+2v+2}}{j-v+1} q_v\right), \qquad j > 1.$$

by use padthey derived eight approximations from this formula (1.6).

In this paper, we will introduce two new approximations from this formula and prove that they represent the upper and lower bounds for Bateman's G –function The calculations and graphs were performed in Wolfram Mathematica.

Preliminaries:

this section we computed new t approximations for $G(x) - \frac{1}{x}$ these approximations were derived using Pad´e approximation. Let $f(x)$ be the formal power series:

$$f(x) = c_0 + c_1 x + c_2 x2 + \cdots.$$

The Pad´e approximant of order (p, q) of $f(x)$ is the rational function $\frac{L(x)}{M(x)}$, where $L(x), M(x)$ are polynomials of degrees $p \geq 0$ and $q \geq 1$, and they denoted by:

$$[p/q]f(x) = \frac{L(x)}{M(x)} = \frac{\sum_{i=0}^{p} a_i x^i}{1+\sum_{i=1}^{q} b_i x^i} \tag{7}$$

where $b1 = \frac{1}{2}$, and:

$$b_k = -2\left(\frac{(2^{2k+2}-1)B_{2k+2}}{k+1} + \sum_{s=1}^{k-1} \frac{(2^{2k-2s+2}-1)B_{2k+2s+2}}{k-s+1} b_s\right), \qquad k > 1.$$

To obtain the Pad´e approximants of order (2,18) and (2,20) of the function$f(x)$andwhich wedenotedbyf_{18} ,f_{20}respectively:

$$f_{18}(x) = \frac{x^{-2}}{u(x)} \tag{8}$$

$$f_{20}(x) = \frac{x^{-2}}{v(x)} \tag{9}$$

Where:

$$u(x) = 2(1 + \sum_{i=1}^{9} b_i x^{-2i} \qquad , v(x) = 2(1 + \sum_{i=1}^{10} b_i x^{-2i}$$

Using Wolfram Mathematica Software ,we obtain:

$$b_2 = -\frac{3}{4}, \quad b_3 = \frac{27}{8}, \quad b_4 = -\frac{423}{16}, b_5 = \frac{9927}{32}, b_6 = -\frac{324423}{64}, b_7 = \frac{14098527}{128}$$

$$b8 = -\frac{787622823}{256}, b_9 = \frac{55075738527}{512} \; and \; b_{10} = -\frac{4716648068823}{1024}$$

Then, we have:

$$u(x) = 2(1 + \frac{1}{2} x^{-2} - \frac{3}{4} x^{-4} + \frac{27}{8} x^{-6} - \frac{423}{16} x^{-8} + \frac{9927}{32} x^{-10} - \frac{324423}{64} x^{-12} + \frac{14098527}{128} x^{-14} - \frac{787622823}{256} x^{16} + \frac{55075738527}{512} x^{-18}).$$

And:

$$v(x) = 2(1 + \frac{1}{2} x^{-2} - \frac{3}{4} x^{-4} + \frac{27}{8} x^{-6} - \frac{423}{16} x^{-8} + \frac{9927}{32} x^{-10} - \frac{324423}{64} x^{-12} + \frac{14098527}{128} x^{-14} - \frac{787622823}{256} x^{-16} + \frac{55075738527}{512} x^{-18} - \frac{4716648068823}{1024} x^{-20})$$

Results and Discussion:

In this section, we deduce two bonds for $G(x)$.To prove the next theorems we need the following auxiliary lemma

Lemma 3.1. *If the function* $g(x)$ *defined for* $x > 0$, $\lim_x \to \infty\, g(x) = 0$ *and* $g(x + \delta) < g(x), \delta > 0$, *then* $g(x) > 0$ *for* $x > 0$ [7].

Theorem 3.2. *The function* $f_{18}(x)$ *is lower bound of* $G(x) - x^{-1}$ *, where the lower bound holds for* $x > 1$.

Proof. *Let* $L(x) = G(x) - \frac{1}{x} - f_{18}(x)$. Then, by using Relation (3.4), we have

$$L(x + 2) - L(x) = -\frac{-18K(x)}{H(x)}$$

Where:

$$\begin{aligned} K(x) = {} & 1475786773533952x^{16} + 23612588376543232x^{15} + 201782758525316096x^{14} \\ & + 1172077432996399104x^{13} + 5065583614598565312x^{12} \\ & + 17028941197493503232x^{11} + 45607551937599249984x^{10} \\ & + 98547862565421190272x^{9} + 172661287706241082192x^{8} \\ & + 245034656119120158336x^{7} + 279737640076416319584x^{6} \\ & + 253417028091835604800x^{5} + 178704623305466581932x^{4} \\ & + 96076527002014840752x^{3} + 22232258848343214892x^{2} \\ & - 21687379028998654728x + 304076448378421989297 \end{aligned}$$

And:

$$\begin{aligned} H(x) = {} & 262144x^{39} + 10223616x^{38} + 189530112x^{37} + 2221146112x^{36} + 18459328512x^{35} \\ & + 115678445568x^{34} + 567400071168x^{33} + 2231772905472x^{32} \\ & + 7154055266304x^{31} + 18889372844032x30 + 41349261312000x^{29} \\ & + 75274167238656x^{28} + 113960881205248x^{27} + 143031107358720x^{26} \\ & + 147858652053504x^{25} + 124764823019520x^{24} + 83665791538176x^{23} \\ & + 39341268077568x^{22} + 61679983959040x^{21} + 43629682725888x^{20} \\ & + 129941579181312x^{19} + 53102469644330752x^{18} + 850064300997932544x^{17} \\ & + 7290733411363335168x^{16} + 42619941990800174016x^{15} \\ & + 185953462404684148800x^{14} + 633499832351415897792x^{13} \\ & + 1727780892184292883264x^{12} + 3825502731787915997136x^{11} \\ & + 6923092782952037084400x^{10} + 10254514580620226229888x^{9} \\ & + 12387993166777199167584x^{8} + 12105584241812549190540x^{7} \\ & + 9453120992790112533204x^{6} + 5664914613779794830900x^{5} \\ & + 2028714909347166450876x^{4} + 2530981074537218075769x^{3} \\ & + 7817843248572163924251x^{2} \\ & + 5473376070811595807346x \qquad > 0, x > 0. \end{aligned}$$

$$\begin{aligned} K(x+1) = {} & 1475786773533952x^{16} + 47225176753086464x^{15} + 733065996997538816x^{14} \\ & + 7302798425066876928x^{13} + 52094480908514481600x^{12} \\ & + 281364328210388290048x^{11} + 1187179814171563865280x^{10} \\ & + 3982711995689176207104x^{9} + 10714528736896043449936x^{8} \\ & + 23151080369744031270144x^{7} + 39980310348387706912800x^{6} \\ & + 54525837164479104739712x^{5} + 57500925563993109137580x^{4} \\ & + 45280426797439363521888x^{3} + 25076592156992722792804x^{2} \\ & + 8677420493971771169424x + 1697901978380645438241 > 0, x > 0 \end{aligned}$$

Furthermore, $lim_x \to \infty\, L(x) = 0, and\ hence, L(x) > 0, \forall\, x > 0, which\ gives\ f(x)_{18} < G(x) - \frac{1}{x}$, $\forall\, x > 1$.

Theorem 3.3.

$The\ function\ f_{20}(x)\ is\ upper\ bound\ of\ G(x) - x^{-1}, where\ the\ upper\ bound\ holds\ for\ x\ > \frac{3}{2}.$

$Proof. Let\ N(x)\ = G(x) - \frac{1}{x} - f_{20}(x). Then, by\ using\ Relation\ (3.4), we\ have$

$$N(x+2) - N(x) = \frac{18S(x)}{x(x+1)(x+2)T(x)W(x)}$$

Where:

$$\begin{aligned} S(x) = {} & 331639705331116032x^{18} + 5969514695960088576x^{17} \\ & + 57761922373616764672x^{16} + 382954758877486870528x^{15} \\ & + 1906508520530498879488x^{14} + 7461590153408242475008x^{13} \\ & + 23558087208296313772096x^{12} + 60908335172258274550528x^{11} \\ & + 130010227082052344163776x^{10} + 229829364573435867372416x^{9} \\ & + 336118213872609075073712x^{8} + 404560336274014057927040x^{7} \\ & + 396938941012996452691616x^{6} + 312874667742612430658240x^{5} \\ & + 190645738222302664492820x^{4} + 78355737923484188154704x^{3} \\ & + 11127563796353333388874132x^{2} + 1934688331706309456070 48x \\ & - 2263536052771366800957057, \end{aligned}$$

$$\begin{aligned} T(x) = {} & 1024x^{20} + 512x^{18} - 768x^{16} + 3456x^{14} - 27072x^{12} + 317664x^{10} - 5190768x^{8} \\ & + 112788216x^{6} - 3150491292x^{4} + 110151477054x^{2} - 4716648068823 \end{aligned}$$

And:

$$\begin{aligned} W(x)\ where, = {} & 1024x^{20} + 40960x^{19} + 778752x^{18} + 9357312x^{17} + 79693056x^{16} + 511352832x^{15} \\ & + 2564877696x^{14} + 10297735680x^{13} + 33609451584x^{12} + 90047096320x^{11} \\ & + 199121068768x^{10} + 364038605184x^{9} + 549269718288x^{8} + 680210383104x^{7} \\ & + 684573380088x^{6} + 551682109344x^{5} + 347270826372x^{4} + 151121810976x^{3} \\ & + 110051647774x^{2} + 369030382456x - 4319131760031 \end{aligned}$$

Where:

$$\begin{aligned} S(x+1) = {} & 1475786773533952x^{16} + 47225176753086464x^{15} + 733065996997538816x^{14} \\ & + 7302798425066876928x^{13} + 52094480908514481600x^{12} \\ & + 281364328210388290048x^{11} + 1187179814171563865280x^{10} \\ & + 3982711995689176207104x^{9} + 10714528736896043449936x^{8} \\ & + 23151080369744031270144x^{7} + 39980310348387706912800x^{6} \\ & + 54525837164479104739712x^{5} + 57500925563993109137580x^{4} \\ & + 45280426797439363521888x^{3} + 25076592156992722792804x^{2} \\ & + 8677420493971771169424x + 1697901978380645438241 \\ & > 0, x > 0, \end{aligned}$$

$$\begin{aligned} T\left(x+\frac{3}{2}\right) = {} & 1024x^{20} + 71680x^{19} + 2383872x^{18} + 50082816x^{17} + 745460928x^{16} + 8356293120x^{15} \\ & + 73195184256x^{14} + 513014194560x^{13} + 2922034290312x^{12} + 13658708280784x^{11} \\ & + 52682701394824x^{10} + 167964753961872x^{9} + \frac{883743342737727x^{8}}{2} \\ & + 953960195115522x^{7} + \frac{3347227244178507x^{6}}{2} + \frac{4698556147679289x^{5}5}{2} \\ & + \frac{164909226173175501x^{4}}{64} + \frac{68102419430857035x^{3}}{32} + \frac{39848128424155511x^{2}}{32} \\ & + \frac{29483124000493975x}{64} + \frac{78953240062810047}{1024} > 0, \quad x > 0 \end{aligned}$$

And:

$$\begin{aligned} W(x+1) = {} & 331639705331116032x^{18} + 11939029391920177152x^{17} \\ & + 209984547120599023360x^{16} + 2389617515056117833728x^{15} \\ & + 19656348080092888987648x^{14} + 123758569623858749382656x^{13} \\ & + 616518065008485767900224x^{12} + 2479048653766145644176896x^{11} \\ & + 8140443109960163701040960x^{10} + 21953264458350291755114752x^{9} \\ & + 48661823406654418689044912x^{8} + 88325331532390279573191424x^{7} \\ & + 130143924131131216816618208x^{6} + 153393623081234503888067200x^{5} \\ & + 141305175527039629436784404x^{4} + 98069149163377190832955552x^{3} \\ & + 48312735240008685504847196x^{2} + 15332268365510617343996016x \\ & + 2148231839564502885753 75 > 0, x > 0, \end{aligned}$$

Furthermore, $lim_x \to \infty\ N(x) = 0, and\ hence, N(x) > 0, \forall\ x > \frac{3}{2}$, which gives $f(x)_{20} > G(x) - \frac{1}{x}, \forall\ x > 1$

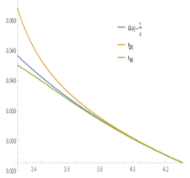

Figure 1: Caption

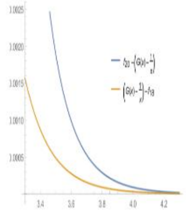

Figure 2: Comparisons between the two approximations

The following inequality holds:

$$f_{18}(x) \; < \; G(x) - \frac{1}{x} \; < \; f_{20}(x),$$

where the lower bound and the upper bound hold for x > 1 and x > 3 2, respectively.

Conclusion:

In this study, we presented new bounds for Bateman's G –function derived from asymptotic form $G(x) \sim \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(2^{2n-}-1)B_{2n}}{nx^{2n}}, \quad x \to \infty$ by using The Pad´e approximant. This study contributes to shedding light on the behavior of new tow functions of the family functions $f(x)_{2n} = \frac{x^{-2}}{2(1+\sum_{i=1}^{n} b_j x^{-2i})}$. We expect that our results will help in utilizing and clarifying this information to study concepts related to pointwise or uniform convergence of sequences of functions for the Bateman's G –function.

Author Contributions:

All the authors read and approved the final version of the paper

Conflicts of Interest:

All the authors declare no conflict of interest.

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