The North African Journal of Scientific Publishing (NAJSP)
مجلة شمال إفريقيا للنشر العلمي (NAJSP)
E-ISSN: 2959-4820
Volume 3, Issue 3, 2025
Page No: 410-418
Website: https://najsp.com/index.php/home/index
SJIFactor 2024: 5.49 | معامل التأثير العربي (AIF) 2024: 0.71 | ISI 2024: 0.696

Logarithmically Weighted K-Contractions in Fréchet Spaces: Existence and Uniqueness For Nonlinear Integral Equations with Fractional Dynamics

Haitham A Makhzoum [1*], Ola A. Mohammed [2]
[1] Department of Mathematics, Faculty of Science, Benghazi University, Benghazi, Libya
[2] Department of Mathematics, Faculty of Science, Omer Al-Mukhtar University, Albeida-Libya

انْكِمَاشَات κ المُوَزَّونَةُ لُوغَارِيثْمِيّاً فِي فَضَاءَات فْرِيشِيه: الوُجُود وَالوَحْدَانِيَّة لِمُعَادَلَات التَّكَامُل غير الخَطِّيَّة ذَات الدّينَامِيكِيَّات الكَسْرِيَّة

هيثم مخزوم[1*]، علا محمد[2]
[1] قسم الرياضيات، كلية العلوم، جامعة بنغازي، بنغازي، ليبيا
[2] قسم الرياضيات، كلية العلوم، جامعة عمر المختار، البيضاء، ليبيا

*Corresponding author: haitham.makhzoum@uob.edu.ly

Received: July 14, 2025 | **Accepted:** September 15, 2025 | **Published:** September 19, 2025

Abstract
This paper presents a novel theoretical framework to establish the existence and uniqueness of solutions for nonlinear functional integral equations. The analysis is conducted within the intricate topological structure of Fréchet spaces, which presents a significant generalization beyond the more common Banach space settings. The central contribution is the formulation of a logarithmically weighted κ-contraction condition, defined as $\kappa := (L_1M_2 + M_1L_2) \cdot e^{-2} < 1$. This specific parameter optimally incorporates the Lipschitz constants L_i, the nonlinear function bounds M_i, and an explicitly derived decay factor associated with the seminorm structure. The methodological paradigm navigates the core topological challenges inherent in non-normable spaces by constructing a convergent sequence of solutions on bounded intervals, and subsequently extending the global solution through a rigorous argument of consistency. The mathematical optimality of the contraction parameter κ is analytically demonstrated by showing that the weight function, expressed as $\ln(1+\tau)/(1+\tau)$, attains its global maximum value of e^{-1} at the specific point where $\tau = e - 1$. The theoretical construct is further validated through a fractional generalization utilizing Mittag-Leffler kernels, and a detailed case study is presented on a physical model of heat transfer that incorporates oscillatory and saturating memory effects. This comprehensive solution theory effectively bridges abstract principles of functional analysis with practical modeling applications for hereditary phenomena in infinite-dimensional systems.

Keywords: Fréchet spaces, Mittag-Leffler kernels, Existence and uniqueness, Nonlinear integral, Urysohn.

الملخص
تتناول هذه الورقة البحثية نتائج شاملة حول الوجود والوحدانية لمعادلات التكامل الدالية غير الخطية في فضاءات فريشيه .وقد تم تقديم إطار عمل جديد قائم على انكماش κ موزون لوغاريتميًا، حيث يُعرَّف شرط الانكماش $\boldsymbol{\kappa} := (\boldsymbol{L_1 M_2} + \boldsymbol{M_1 L_2}) \cdot \boldsymbol{e^{-2}} < \mathbf{1}$ بما يحقق تزامنًا منهجيًا بين ثوابت ليبشيتز $\boldsymbol{L_i}$، والحدود غير الخطية $\boldsymbol{M_i}$، وانحسار أشباه المعايير الأمثل .تعالج المنهجية التحديات الطوبولوجية الأساسية في الفضاءات غير القابلة للقياس المعياري (non-normable spaces) من خلال بناء انكماشات متتابعة على فترات منتهية، ثم تمديد الحلول

بشكل عالمي عبر الاستمرار المتسق .كما طُوِّر امتداد كسري باستخدام نوى ميتاغ-ليفلر (Mittag-Leffler kernels) مع تطبيق فيزيائي مفصل على انتقال الحرارة ذي التأثيرات الذاكرية المتذبذبة-المشبعة، بما يؤكد الإطار النظري بشكل صارم .وقد تبيّن أن معامل الانكماش $\boldsymbol{\kappa}$ ذو حدة تحليلية قصوى من خلال دراسة متطرفة (extremal analysis) لدالة الوزن $\mathbf{ln(1+t)/(1+t)}$, :حيث تحقق القيمة العظمى $\boldsymbol{e^{-1}}$ **at** $\boldsymbol{t = e - 1}$ إن النظرية الكاملة للحلول التي تم التوصل إليها تجسر الفجوة بين التحليل الدالي التجريدي والنمذجة العملية للظواهر الوراثية في فضاءات لانهائية الأبعاد.

الكلمات المفتاحية: فضاءات فريشيه، نوى ميتاغ-ليفلر، الوجود والوحدانية، معادلات التكامل غير الخطية، أوريشون.

Introduction

In the field of mathematical modeling, nonlinear functional integral equations are pivotal instruments for representing complex systems that possess memory effects. Such systems are ubiquitous across a multitude of scientific disciplines, including but not limited to viscoelasticity, thermal transport, and various forms of anomalous diffusion [1, 2]. A robust theoretical foundation for the existence and uniqueness of solutions to these equations has been extensively developed within Banach spaces, primarily through the application of classical fixed-point theorems [3, 4, 5]. However, the generalization of these analytical constructs to the more abstract and topologically intricate domain of Fréchet spaces poses profound theoretical complexities and unique challenges [6].

Within these infinite-dimensional topological vector spaces, the underlying structure is not defined by a single norm but rather by a countably infinite family of seminorms. This fundamental characteristic imposes significant constraints on the direct application of traditional contraction principles, particularly when one considers unbounded temporal or spatial domains. Various scholarly endeavors have investigated these intricate complexities, employing different methodologies such as compactness arguments or generalized fixed-point frameworks to analyze equations of the Urysohn type [7, 8]. These prior investigations underscore the sophisticated analytical obstacles that arise in the absence of a normable structure and emphasize the critical need for novel methodological approaches capable of ensuring uniform convergence behavior across the entire spectrum of seminorms concurrently.

Building upon this established corpus of literature, the current investigation addresses a nonlinear functional integral equation characterized by a product-type nonlinearity, a structural formulation previously examined using the Schauder fixed-point theorem [9]. The equation is expressed as follows:

$$z(\tau) = c(\tau) + \left(\int_0^\tau g_1(\tau, \sigma, z(\sigma))d\sigma\right)\left(\int_0^\tau g_2(\tau, \sigma, z(\sigma))d\sigma\right), \tau \geq 0 \qquad (1)$$

This specific mathematical form naturally appears in physical models that involve the confluence of hereditary memory mechanisms with nonlinear multiplicative interactions. To establish a unified theoretical framework for existence and uniqueness, a logarithmically weighted κ-contraction paradigm is introduced. The central contraction constraint, formulated as $\kappa := (L_1M_2 + M_1L_2) \cdot e^{-2} < 1$, functions to systematically integrate the Lipschitz constants (L_i), the nonlinear bounds (M_i), and an analytically derived optimal seminorm decay factor. The present rigorous analysis demonstrates that this decay structure is mathematically sharp, as it is derived from the global maximum of the weight function $\frac{\ln(1+\tau)}{1+\tau}$, which achieves its peak value of e^{-1} precisely at the temporal coordinate $\tau = e - 1$.

The principal mathematical contributions of this research encompass the development of a unified fixed-point framework for nonlinear functional integral equations in Fréchet spaces [10], and the provision of a fractional extension that utilizes Mittag-Leffler kernels, which possesses direct applicability to models of anomalous diffusion and the behavior of viscoelastic materials [11, 12]. Additionally, a rigorous physical validation is provided through a comprehensive application to a heat transfer model that incorporates oscillatory-saturating memory effects, thereby confirming the practical utility of the theoretical results. This exhaustive solution theory forges a direct link between abstract principles of functional analysis and concrete modeling of hereditary phenomena, consequently expanding the analytical toolkit available for investigating nonlinear integral equations.

Preliminary Concepts and Definitions

Topological Vector Spaces and Contraction Mappings.

Definition 1 (Fréchet Space). A topological vector space is classified as a Fréchet space if it is locally convex, possesses a metrizable topology, and is complete with respect to that topology. These spaces are characterized by a countable family of seminorms, denoted by $\{\|\cdot\|_\ell\}_{\ell\in\mathbb{N}}$. In this particular work, the focus is placed upon the function space $\mathcal{Z} = C([0,\infty),\mathbb{R})$, which consists of all continuous real-valued functions defined on the non-negative real line. The topological structure of this space is induced by the following family of seminorms:

$$\|z\|_\ell = \sup_{\tau\in[0,\ell]} |z(\tau)|, \forall \ell \in \mathbb{N}$$

The space $\mathcal{Z}$ is known to be complete under this collection of seminorms. This completeness arises from the fact that uniform convergence on every compact interval $[0,\ell]$ ensures the preservation of continuity. Thus, any sequence that exhibits Cauchy behavior with respect to each seminorm $\|\cdot\|_\ell$ will necessarily converge to a function that maintains its continuity across the entire domain $[0,\infty)$.

Definition 2 (Sequential Contraction). Consider the Fréchet space $\mathcal{Z} = C([0,\infty),\mathbb{R})$ endowed with the seminorms $\|z\|_\ell = \sup_{\tau\in[0,\ell]} |z(\tau)|$. An operator $\mathcal{T}: \mathcal{Z} \to \mathcal{Z}$ is formally classified as a sequential Contraction under the simultaneous satisfaction of the following three conditions:

Local Contraction Property: For every positive integer $\ell \in \mathbb{N}$, the operator $\mathcal{T}$ constitutes a strict κ-contraction when its domain is restricted to the Banach space ($C([0,\ell]), \|\cdot\|_\ell$):

$$\|\mathcal{T}(z_1) - \mathcal{T}(z_2)\|_\ell \le \kappa\|z_1 - z_2\|_\ell, \forall z_1, z_2 \in \mathcal{Z} \text{ where } 0 \le \kappa < 1$$

Solution Consistency: Let z_ℓ be a solution to the operator equation $z_\ell = \mathcal{T}(z_\ell)$ within the restricted space $C([0,\ell])$. Then the following consistency relation must be maintained:

$$z_\ell(\tau) = z_{\ell'}(\tau)\ \forall \tau \in [0,\ell'], \forall \ell' < \ell$$

Global Convergence: The pointwise limit $z(\tau) = \lim_{\ell\to\infty} z_\ell(\tau)$ exists within the space $\mathcal{Z}$ and satisfies the operator equation $\mathcal{T}(z) = z$ globally across the entire domain $[0,\infty)$.

Definition 3 (Lipschitz Condition with Attenuation). A function $g: [0,\infty)^2 \times \mathbb{R} \to \mathbb{R}$ is said to satisfy a time-attenuating Lipschitz condition if there exists a positive constant $L > 0$ such that the following inequality holds universally:

$$|g(\tau,\sigma,z_1) - g(\tau,\sigma,z_2)| \le \frac{L}{(1+\tau)(1+\sigma)}|z_1 - z_2|, \forall \tau,\sigma \in [0,\infty), \forall z_1, z_2 \in \mathbb{R}$$

This particular decay structure serves to mathematically model the fading memory phenomenon commonly observed in various physical systems.

Definition 4 (Mittag-Leffler Kernel for Fractional Systems). The Mittag-Leffler function, denoted as $E_{\alpha,\beta}(\zeta)$, is defined by its convergent series expansion [12]:

$$E_{\alpha,\beta}(\zeta) = \sum_{\kappa=0}^{\infty} \frac{\zeta^\kappa}{\Gamma(\alpha\kappa+\beta)}, \alpha > 0, \beta > 0$$

For the purpose of analyzing fractional dynamical systems, a memory kernel exhibiting temporal decay is considered, taking the following specific form:

$$\mathcal{K}(\tau,\sigma) = \frac{E_{\alpha,1}(-\lambda(\tau-\sigma)^\alpha)}{(1+\tau)(1+\sigma)}, \lambda > 0, \alpha \in (0,1)$$

The subsequent fundamental assumptions constitute the basis of the mathematical analysis presented in this work:

The function $c \in \mathcal{Z}$ is a continuous and bounded function defined over the entire interval $[0, \infty)$. For each index $i = 1,2$, the function g_i exhibits continuity with respect to the variable τ and satisfies a time-attenuating Lipschitz condition with an associated constant L_i :

$$|g_i(\tau, \sigma, z_1) - g_i(\tau, \sigma, z_2)| \leq \frac{L_i}{(1+\tau)(1+\sigma)} |z_1 - z_2|.$$

The nonlinear terms are uniformly bounded, subject to the following inequality:

$$|g_i(\tau, \sigma, z)| \leq \frac{M_i}{(1+\tau)(1+\sigma)}, \forall \tau, \sigma \in [0, \infty), \forall z \in \mathbb{R}.$$

The following supremum condition is required to hold across all-natural numbers:

$$\sup_{\ell \in \mathbb{N}} \left(\|c\|_\ell + \frac{M_1 M_2}{e^2} \right) < \infty.$$

Theorem (Leray-Schauder Alternative [13]). Let $\mathcal{T}$ be a contraction mapping defined on a closed Subset $\mathcal{W} \subset \mathcal{Z}$, and suppose that the image set $\mathcal{T}(\mathcal{W})$ is bounded. Then one of the following mutually exclusive statements must hold:

The operator $\mathcal{T}$ possesses a unique fixed point within $\mathcal{W}$, or There exist a scalar $\lambda \in (0,1)$, a natural number $\ell \in \mathbb{N}$, and a function $u \in \partial \mathcal{W}_\ell$ such that the following condition is satisfied:

$$\|u - \lambda \mathcal{T}(u)\|_\ell = 0.$$

Main Result

Formulation and Operator Continuity.

The central concern of this investigation is the nonlinear functional integral equation with a product structure, previously introduced as:

$$z(\tau) = c(\tau) + \left(\int_0^\tau g_1(\tau, \sigma, z(\sigma)) d\sigma \right) \left(\int_0^\tau g_2(\tau, \sigma, z(\sigma)) d\sigma \right), \tau \in [0, \infty). \tag{2}$$

In this context, $c: [0, \infty) \to \mathbb{R}$ is a continuous function, and the functions $g_i: [0, \infty)^2 \times \mathbb{R} \to \mathbb{R}$ for $i = 1,2$ are given functions that satisfy the comprehensive assumptions (A1)-(A4).

An associated nonlinear operator $\mathcal{T}: \mathcal{Z} \to \mathcal{Z}$ is defined as follows:

$$\mathcal{T}(z)(\tau) = c(\tau) + \left(\int_0^\tau g_1(\tau, \sigma, z(\sigma)) d\sigma \right) \left(\int_0^\tau g_2(\tau, \sigma, z(\sigma)) d\sigma \right). \tag{3}$$

Lemma 1 (Continuity of the Operator). Given that the assumptions (A1)-(A4) hold, the operator $\mathcal{T}: \mathcal{Z} \to \mathcal{Z}$ is well-defined and demonstrates continuity within the Fréchet topological structure. Proof. To establish the well-definedness of the operator, it must be verified that for an arbitrary function $z \in \mathcal{Z}$, the resulting function $\mathcal{T}(z)$ maintains continuity on $[0, \infty)$. Let an arbitrary point $\tau_0 \geq 0$ be selected. The difference $|\mathcal{T}(z)(\tau) - \mathcal{T}(z)(\tau_0)|$ can be bounded from above by the following expression:

$$|\mathcal{T}(z)(\tau) - \mathcal{T}(z)(\tau_0)| \leq |c(\tau) - c(\tau_0)| + \left| \left(\prod_{i=1}^{2} \mathcal{J}_i(\tau) \right) - \left(\prod_{i=1}^{2} \mathcal{J}_i(\tau_0) \right) \right|,$$

where $\mathcal{J}_i(\tau) = \int_0^\tau g_i(\tau, \sigma, z(\sigma)) d\sigma$. Assumption (A1) ensures that the function c is continuous. For the product term, the following inequality can be employed:

$$\left| \prod \mathcal{J}_i(\tau) - \prod \mathcal{J}_i(\tau_0) \right| \leq \sum_{j=1}^{2} |\mathcal{J}_j(\tau) - \mathcal{J}_j(\tau_0)| \prod_{k \neq j} \sup_{\tau' \in [0,\infty)} |\mathcal{J}_k(\tau')|.$$

By applying assumption (A3) and considering the integral bound $\int_0^\tau \frac{d\sigma}{(1+\sigma)} = \ln(1+\tau)$, the integral terms are bounded as follows:

$$|\mathcal{J}_i(\tau)| \le M_i \frac{\ln(1+\tau)}{1+\tau} \le M_i e^{-1}.$$

To demonstrate the continuity of $\mathcal{J}_i(\tau)$, one observes that the integrand $g_i(\tau, \sigma, z(\sigma))$ is continuous with respect to τ for a fixed σ (by (A2)). It is also dominated by the integrable function $\frac{M_i}{(1+\sigma)^2}$ for $\sigma \in [0, \tau]$ and for τ within any neighborhood of τ_0 (as a consequence of (A3)). Consequently, an application of the

Dominated Convergence Theorem guarantees the continuity of $\mathcal{J}_i(\tau)$ at τ_0. This line of reasoning leads to the conclusion that the operator output $\mathcal{T}(z)$ is a continuous function.

To prove continuity in the Fréchet topology, for each $\ell \in \mathbb{N}$, continuity must be established with respect to the seminorm $\|\cdot\|_\ell$. Let the sequence of functions $\{z_k\}$ converge to z in $\mathcal{Z}$, which is equivalent to uniform convergence on every compact interval $[0, \ell]$. One may then consider the difference $\|\mathcal{T}(z_k) - \mathcal{T}(z)\|_\ell$, which is bounded by:

$$\|\mathcal{T}(z_k) - \mathcal{T}(z)\|_\ell \le \|c - c\|_\ell + \left\| \left(\prod \mathcal{J}_i(z_k) \right) - \left(\prod \mathcal{J}_i(z) \right) \right\|_\ell$$

The product term satisfies the following bound:

$$\left\| \left(\prod \mathcal{J}_i(z_k) \right) - \left(\prod \mathcal{J}_i(z) \right) \right\|_\ell \le \sum_{j=1}^{2} \|\mathcal{J}_j(z_k) - \mathcal{J}_j(z)\|_\ell \prod_{m \ne j} \sup_{\tau \in [0,\ell]} |\mathcal{J}_m(z_k(\tau))|$$

Utilizing assumptions (A2) and (A3) and noting that for $\tau \in [0, \ell]$, $\frac{\ln(1+\tau)}{1+\tau} \le e^{-1}$, one can bound the difference between the integral terms:

$$\left|\mathcal{J}_j(z_k)(\tau) - \mathcal{J}_j(z)(\tau)\right| \le \int_0^\tau \frac{L_j}{(1+\tau)(1+\sigma)} |z_k(\sigma) - z(\sigma)| d\sigma \le L_j \frac{\ln(1+\tau)}{1+\tau} \|z_k - z\|_\ell \le L_j e^{-1} \|z_k - z\|_\ell.$$

As $k \to \infty$, it follows that $\|z_k - z\|_\ell \to 0$, which in turn implies $\left\|\mathcal{J}_j(z_k) - \mathcal{J}_j(z)\right\|_\ell \to 0$. Given that the supremum of the integral terms, $\sup_{\tau \in [0,\ell]} |\mathcal{J}_m(z_k(\tau))|$, is bounded by $M_m e^{-1}$, it is therefore established that $\|\mathcal{T}(z_k) - \mathcal{T}(z)\|_\ell \to 0$ as $k \to \infty$.

Lemma 2 (Global Solution via Sequential Contraction). Suppose that the operator $\mathcal{T}$ fulfills both the local contraction property (i) and the solution consistency property (ii) from Definition 2.2. Under these conditions, the sequence $\{z_\ell\}$ of fixed points obtained on the finite intervals $[0, \ell]$ converges uniformly on all compact subsets of $[0, \infty)$ to a global solution $z \in \mathcal{Z}$.

Proof. By virtue of property (i), for each positive integer ℓ, there exists a unique fixed point $z_\ell \in C([0, \ell])$ that satisfies the relationship $z_\ell = \mathcal{T}(z_\ell)$. The solution consistency condition (ii) provides the compatibility relation $z_\ell|_{[0,\ell']} = z_{\ell'}$ for all $\ell' < \ell$. This inherent consistency enables the definition of a global function $z: [0, \infty) \to \mathbb{R}$ by setting $z(\tau) = z_\ell(\tau)$ for any $\ell > \tau$. The function z is well-defined and continuous on $[0, \infty)$ since its restriction to any compact interval $[0, \ell]$ is continuous. Furthermore, for any $\tau \ge 0$, by selecting an integer $\ell > \tau$, one has $z(\tau) = z_\ell(\tau) = \mathcal{T}(z_\ell)(\tau) = \mathcal{T}(z)(\tau)$, as the operator $\mathcal{T}$ depends solely on the values of its argument up to τ, and $z(\sigma) = z_\ell(\sigma)$ for all $\sigma \in [0, \tau]$. Consequently, it is demonstrated that z is a fixed point of $\mathcal{T}$ in the space $\mathcal{Z}$.

To substantiate that the sequence $\{z_\ell\}$ converges to z uniformly on any compact interval $[0, \ell']$, it is sufficient to note that for any $\ell > \ell'$, the restriction $z_\ell|_{[0,\ell']}$ is identical to $z_{\ell'}$ due to the consistency property. This implies that on the interval $[0, \ell']$, the sequence $\{z_\ell\}$ becomes constant for all $\ell > \ell'$ and is equal to $z_{\ell'} = z|_{[0,\ell']}$. As a result, the convergence is uniform on any compact interval of interest.

Theorem (Existence and Uniqueness). Assuming that $c \in \mathcal{Z}$ and conditions (A1)-(A4) are fulfilled, if the contraction parameter $\kappa := (L_1 M_2 + M_1 L_2) \cdot e^{-2} < 1$, then:

The operator $\mathcal{T}$ defined in (3) constitutes a sequential contraction (as per Definition 2.2). The functional equation (2) possesses a unique solution within the space $\mathcal{Z}$.

Proof. First, it must be established that the operator $\mathcal{T}$ satisfies the sequential contraction properties.

Let $z_1, z_2 \in \mathcal{Z}$ be arbitrary functions and fix a value $\tau \in [0, \ell]$. The difference $|\mathcal{T}(z_1)(\tau) - \mathcal{T}(z_2)(\tau)|$ may be estimated by adding and subtracting appropriate terms:

$$|\mathcal{T}(z_1)(\tau) - \mathcal{T}(z_2)(\tau)| \leq \left|\int_0^\tau g_1(\tau,\sigma,z_1(\sigma))d\sigma\right| \cdot \left|\int_0^\tau [g_2(\tau,\sigma,z_1(\sigma)) - g_2(\tau,\sigma,z_2(\sigma))]d\sigma\right| + \left|\int_0^\tau [g_1(\tau,\sigma,z_1(\sigma)) - g_1(\tau,\sigma,z_2(\sigma))]d\sigma\right| \cdot \left|\int_0^\tau g_2(\tau,\sigma,z_2(\sigma))d\sigma\right|$$

Upon applying the boundedness (A3) and Lipschitz (A2) conditions, and using the bound $\|z_1 - z_2\|_\ell$ for the seminorm on $[0, \ell]$, the inequality becomes:

$$\leq \left(\int_0^\tau \frac{M_1}{(1+\tau)(1+\sigma)}d\sigma\right) \cdot \left(\int_0^\tau \frac{L_2}{(1+\tau)(1+\sigma)}\|z_1 - z_2\|_\ell d\sigma\right) + \left(\int_0^\tau \frac{L_1}{(1+\tau)(1+\sigma)}\|z_1 - z_2\|_\ell d\sigma\right) \cdot \left(\int_0^\tau \frac{M_2}{(1+\tau)(1+\sigma)}d\sigma\right)$$

$$= \left(M_1 \frac{\ln(1+\tau)}{1+\tau}\right)\left(L_2 \frac{\ln(1+\tau)}{1+\tau}\|z_1 - z_2\|_\ell\right) + \left(L_1 \frac{\ln(1+\tau)}{1+\tau}\|z_1 - z_2\|_\ell\right)\left(M_2 \frac{\ln(1+\tau)}{1+\tau}\right)$$

$$= (M_1L_2 + L_1M_2)\frac{[\ln(1+\tau)]^2}{(1+\tau)^2}\|z_1 - z_2\|_\ell$$

The next step is to determine the maximum value of the function $\zeta(\tau) = \frac{\ln(1+\tau)}{1+\tau}$ for $\tau \geq 0$. Differentiation of this function yields:

$$\zeta'(\tau) = \frac{(1+\tau) \cdot \frac{1}{1+\tau} - \ln(1+\tau) \cdot 1}{(1+\tau)^2} = \frac{1 - \ln(1+\tau)}{(1+\tau)^2}.$$

Setting $\zeta'(\tau) = 0$ provides the condition $\ln(1+\tau) = 1$, which implies that the critical point occurs at $\tau = e - 1$. The second derivative confirms that this point is a maximum:

$$\zeta''(\tau) = \frac{-3 + 2\ln(1+\tau)}{(1+\tau)^3}.$$

At the point $\tau = e - 1, \zeta''(e-1) = -1/e^3 < 0$, which confirms a global maximum. Consequently, the global maximum value is $\zeta(e-1) = e^{-1}$. From this reasoning, it follows that:

$$\frac{[\ln(1+\tau)]^2}{(1+\tau)^2} \leq e^{-2}\ \forall \tau \geq 0$$

By substituting this bound back into the estimation, it is found that:

$$|\mathcal{T}(z_1)(\tau) - \mathcal{T}(z_2)(\tau)| \leq (M_1L_2 + L_1M_2) \cdot e^{-2}\|z_1 - z_2\|_\ell = \kappa\|z_1 - z_2\|_\ell$$

Taking the supremum over $\tau \in [0, \ell]$ yields $\|\mathcal{T}(z_1) - \mathcal{T}(z_2)\|_\ell \leq \kappa\|z_1 - z_2\|_\ell$. Since the condition $\kappa < 1$ is given, the operator $\mathcal{T}$ is a contraction on $(C([0,\ell]), \|\cdot\|_\ell)$ for each integer ℓ.

Let z_ℓ be the unique solution to $z_\ell = \mathcal{T}(z_\ell)$ on the interval $[0, \ell]$, and let $z_{\ell'}$ be the solution on $[0, \ell']$ for $\ell' < \ell$. The function $z_\ell|_{[0,\ell']}$ represents the restriction of z_ℓ to the smaller interval $[0, \ell']$. For any $\tau \in [0, \ell']$, the following expression holds:

$$z_\ell|_{[0,\ell']}(\tau) = z_\ell(\tau) = \mathcal{T}(z_\ell)(\tau) = c(\tau) + \left(\int_0^\tau g_1(\tau,\sigma,z_\ell(\sigma))d\sigma\right)\left(\int_0^\tau g_2(\tau,\sigma,z_\ell(\sigma))d\sigma\right)$$

Since the integrals depend only on the values of $z_\ell(\sigma)$ for $\sigma \in [0,\tau] \subset [0,\ell']$, this may be expressed as:

$$z_\ell|_{[0,\ell']}(\tau) = \mathcal{T}\left(z_\ell|_{[0,\ell']}\right)(\tau).$$

This demonstrates that $z_\ell|_{[0,\ell']}$ is a fixed point of the operator $\mathcal{T}$ within the space $C([0,\ell'])$. Given that the Banach fixed-point theorem ensures a unique fixed point in this space, it must be the case that $z_\ell|_{[0,\ell']} = z_{\ell'}$. This reasoning establishes the solution consistency property.

As it has been proven that $\mathcal{T}$ satisfies both the local contraction and the solution consistency properties, Lemma guarantees the existence of a global solution $z \in \mathcal{Z}$ to the equation $z = \mathcal{T}(z)$.

To demonstrate uniqueness, an assumption can be made, for the sake of contradiction, that z_a and z_b are two distinct global solutions within $\mathcal{Z}$. This implies that for any $\ell \in \mathbb{N}$, their respective restrictions $z_a|_{[0,\ell]}$ and $z_b|_{[0,\ell]}$ are both fixed points of $\mathcal{T}$ within the Banach space $C([0,\ell])$. The uniqueness of the fixed point in this space, as guaranteed by the Banach fixed-point theorem, implies that $z_a|_{[0,\ell]} = z_b|_{[0,\ell]}$. Since this equality holds for all ℓ, it must be that $z_a(\tau) = z_b(\tau)$ for all $\tau \geq 0$, which contradicts the initial assumption that they are distinct solutions.

Corollary **(Fractional Extension with Mittag-Leffler Kernels)**. Consider a fractional integral equation with inherent memory effects:

$$z(\tau) = c(\tau) + \prod_{i=1}^{2} \left(\int_0^\tau \frac{E_{\alpha_i,1}(-\lambda_i(\tau-\sigma)^{\alpha_i})}{(1+\tau)(1+\sigma)} g_i(\sigma, z(\sigma)) d\sigma \right) \tag{4}$$

where $E_{\alpha_i,1}$ represents the Mittag-Leffler function. It can be shown that Theorem holds, provided that the modified contraction condition.

$$\kappa^\alpha := \kappa \cdot \max_{i=1,2} \sup_{\tau \geq 0} \left(\frac{\left| \int_0^\tau E_{\alpha_i,1}(-\lambda_i(\tau-\sigma)^{\alpha_i}) \frac{d\sigma}{1+\sigma} \right|}{1+\tau} \right) < 1$$

is satisfied, where κ is defined as in Theorem. A sufficient condition for this is $\kappa^\alpha := \kappa \cdot \max_{i=1,2} |$ $E_{\alpha_i,1}(-\lambda_i(e-1)^{\alpha_i})$ $|< 1$, assuming that the Mittag-Leffler function's maximum value is attained near the critical point of the logarithmic weight.

Comprehensive Application: A Nonlinear Heat Transfer Model

A physically motivated heat transfer process that includes nonlinear dissipation and memory is considered here. The model can be expressed as:

$$z(\tau) = e^{-\tau} + \left(\int_0^\tau \frac{\sin(z(\sigma))}{e^{\tau+\sigma}} d\sigma \right) \left(\int_0^\tau \frac{\arctan(z(\sigma))}{e^{\tau+\sigma}} d\sigma \right), \tau \geq 0 \tag{5}$$

In this formulation, the term $e^{-\tau}$ serves as a representation of ambient heat loss, the term $\sin(z(\sigma))$ captures oscillatory thermal interactions, and the term $\arctan(z(\sigma))$ models flux saturation. The kernel $e^{-(\tau+\sigma)}$ signifies an exponential attenuation of historical contributions over time.

Verification of Assumptions. The first function, $\sin(z)$, has a Lipschitz constant of 1, as the Mean Value Theorem gives $|\sin(z_a) - \sin(z_b)| \leq |z_a - z_b|$. Similarly, for the second function, $\arctan(z)$, its derivative is $\frac{1}{1+z^2}$, which is bounded by 1. Consequently, one obtains the Lipschitz constants $L_1 = 1$ and $L_2 = 1$. For the nonlinear terms, the following uniform bounds are established:

$$\left| \frac{\sin(z)}{e^{\tau+\sigma}} \right| \leq \frac{1}{e^{\tau+\sigma}} \leq \frac{1}{(1+\tau)(1+\sigma)}$$

And

$$\left| \frac{\arctan(z)}{e^{\tau+\sigma}} \right| \leq \frac{\pi/2}{e^{\tau+\sigma}} \leq \frac{\pi/2}{(1+\tau)(1+\sigma)}$$

Thus, the uniform bounds are found to be $M_1 = 1$ and $M_2 = \pi/2 \approx 1.5708$.

Sequential contraction parameter: The parameter κ is computed as:

$$\kappa = \left(1 \cdot \frac{\pi}{2} + 1 \cdot 1\right) \cdot \frac{1}{e^2} \approx (1.5708 + 1) \cdot 0.1353 \approx 0.348 < 1$$

Since all the required assumptions are fully met, Theorem guarantees the existence of a unique global solution for equation (5).

Fractional Extension to Viscoelastic Materials.

The model is now extended to a fractional form, applicable for viscoelastic materials that exhibit power-law memory:

$$z(\tau) = e^{-\tau} + \left(\int_0^\tau \frac{E_{0.8,1}(-(\tau-\sigma)^{0.8})}{e^{\tau+\sigma}} \sin(z(\sigma))d\sigma\right)\left(\int_0^\tau \frac{E_{0.8,1}(-(\tau-\sigma)^{0.8})}{e^{\tau+\sigma}} \arctan(z(\sigma))d\sigma\right) \quad (6)$$

The modified contraction parameter in this case incorporates the decay characteristic of the MittagLeffler function. By applying the sufficient condition from the corollary, it is found that:

$$\kappa^{0.8} = \kappa \cdot \left|E_{0.8,1}(-(e-1)^{0.8})\right| \approx 0.348 \cdot 0.487 \approx 0.169 < 1$$

This result confirms that the uniqueness of the solution is preserved, in accordance with the findings of Corollary.

Conclusion

This academic paper has successfully developed a comprehensive and robust mathematical paradigm for the resolution of nonlinear functional integral equations within the topological framework of Fréchet spaces. A novel logarithmically weighted κ-contraction construct was introduced, which systematically aligns Lipschitz constants, nonlinear bounds, and a seminorm decay factor to guarantee consistent convergence across all seminorms simultaneously. The rigorous derivation of the logarithmic decay factor, which precisely demonstrates that the expression $\frac{[\ln(1+\tau)]^2}{(1+\tau)^2}$ is bounded by e^{-2}, coupled with a proof of operator continuity (Lemma), serves to complete the theoretical foundations. From this perspective, the solution defined as $z(\tau) = \lim_{\ell\to\infty} z_\ell(\tau)$ is now fully justified as a global solution within the space $\mathcal{Z}$.

Furthermore, it was demonstrated that the κ-contraction framework can be naturally extended to incorporate Mittag-Leffler kernels, thereby providing a unified solution theory for equations that model complex memory effects in various physical systems. The provided heat transfer case study serves as a rigorous validation of this theoretical paradigm, demonstrating meticulous adherence to all mathematical assumptions and confirming the concrete applicability of the methodology to real-world dynamical systems characterized by hereditary memory.

By bridging abstract principles of functional analysis with practical mathematical modeling, this work makes a significant contribution to the analytical toolkit available for studying nonlinear phenomena in engineering and physics. The results conclusively establish that analytical techniques developed within the more restrictive context of finite-dimensional spaces can be rigorously and successfully extended to infinite-dimensional spaces while maintaining complete mathematical integrity. This opens up new avenues for future research into complex systems with hereditary characteristics.

References

[1] P. J. Collins, Differential and Integral Equations. OUP Oxford, 2006.

[2] B. G. Pachpatte, "On a nonlinear Volterra-Fredholm integral equation," Sarajevo J. Math, vol. 16, pp. 61-71, 2008.

[3] E. Zeidler, Nonlinear Functional Analysis and Its Applications I. Springer Verlag, 1985.

[4] S. Althubiti, H. A. Makhzoum, and Y. N. Raffoul, "Periodic solution and stability in nonlinear neutral system with infinite delay," Appl. Math. Sci, vol. 7, no. 136, pp. 6749-6764, 2013.

[5] A. A. B. Fayed, H. A. Makhzoum, R. A. Elmansouri, and A. K. Alshaikhly, "Periodic solutions by Krasnoselskii fixed point theorem of neutral nonlinear system of dynamical equation with Variable Delays," Int. J. Sci. Res. in Mathematical and Statistical Sciences, vol. 8, no. 5, 2023.

[6] H. A. Makhzoum, I. F. Ben Saoud, and R. A. Elmansouri, "On Solutions of nonlinear functional integral equations in Frechet space," Int. J. Sci. Res. in Mathematical and Statistical Sciences, vol. 10, no. 1, pp. 101-110, 2022.

[7] M. Benchohra and M. A. Darwish, "On quadratic integral equations of Urysohn type in Fréchet spaces," Acta Mathematica Universitatis Comenianae. New Series, vol. 79, no. 1, pp. 105-110,

2010.
[8] L. H. Sadoon, "On Solutions of a Quadratic Integral Equation of Urysohn Type," Journal of Education and Science, vol. 27, no. 1, pp. 92-96, 2014.
[9] I. F. BenSaoud, H. A. Makhzoum, and K. M. Msaikc, "Existence of at least one positive continuing solution of Urysohn quadratic integral equation by Schauder fixed-point theorem," Libyan Journal of Science Technology, vol. 13, no. 2, pp. 112-115, 2021.
[10] H. A. Makhzoum, A. S. Elmabrok, and R. A. Elmansouri, "The Existence of a Unique Solution of Nonlinear Integral Equation in Fréchet Space," Global Libyan Journal, vol. 59, 2022.
[11] R. W. Ibrahim and S. Momani, "Upper and lower bounds of solutions for fractional integral equations," Surveys in Mathematics and its Applications, vol. 2, pp. 145-156, 2007.
[12] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. San Diego: Academic Press, 1998.
[13] A. Granas and J. Dugundji, Fixed Point Theory. New York: Springer-Verlag, 2003.