The North African Journal of Scientific Publishing (NAJSP)
مجلة شمال إفريقيا للنشر العلمي (NAJSP)
E-ISSN: 2959-4820
Volume 3, Issue 3, 2025
Page No: 215-220
Website: https://najsp.com/index.php/home/index
SJIFactor 2024: 5.49 | معامل التأثير العربي (AIF) 2024: 0.71 | ISI 2024: 0.696

The Stone-Cech Compactification Of Fibrewise Topological Spaces

Najia Al-Sharrani*
Department of Mathematics, Faculty of Science, Al-Asmariya Islamic University, Zliten, Libya

دمج ستون-تشيك للمساحات الطوبولوجية الليفية

نجية الشراني*
قسم الرياضيات، كلية العلوم، الجامعة الأسمرية الإسلامية، زليتن، ليبيا

*Corresponding author: n.alsharrani@asmarya.edu.ly

Received: July 02, 2025	Accepted: August 05, 2025	Published: August 14, 2025

Abstract:
The purpose of this paper is to introduce and investigate new concepts related to fibrewise topological spaces over a base space C. We begin by defining what constitutes a fibrewise topological space and examining its key properties. We then address the notions of compactness and compactification within this context. In particular, we demonstrate that every fibrewise continuous function from A into any fibrewise compact space can be uniquely extended to the largest compactification of A. Several additional results concerning fibrewise compactification are also presented and discussed.

Keywords:Fibrewise topological space, Fibrewise compact space, Fibrewise compactification, Stone-Cech compactification of fibrewise, Topological spaces.

الملخص
تهدف هذه الورقة البحثية إلى طرح ودراسة مفاهيم جديدة تتعلق بالفضاءات الطوبولوجية الليفية على فضاء القاعدة C. نبدأ بتعريف ماهية الفضاء الطوبولوجي الليفي ودراسة خصائصه الرئيسية. ثم نتناول مفهومي التماسك والتماسك في هذا السياق. ونوضح تحديدًا أن كل دالة ليفية متصلة من A إلى أي فضاء ليفي مضغوط يمكن توسيعها بشكل فريد إلى أكبر تماسك لـ A. كما نعرض ونناقش العديد من النتائج الإضافية المتعلقة بالتماسك الليفي.

الكلمات المفتاحية: الفضاء التبولوجي الليفي، الفضاء التبولوجي الليفي المتراص، الدمج (التمكيث) الليفي دمج (تمكيث) الستون-تشيك للفضاءات التبولوجية الليفية.

Introduction

Fibrewise topological spaces theory, presented in the recent 20 years, as a new branch of mathematics developed based on General Topology, Algebraic Topology. It is associated with differential geometry, lie groups and dynamical systems theory. From the perspective of category theory, it is in the higher category of general topological spaces, so the discussion of new properties and characteristics of the variety of fibre topological spaces has more important significance [1].

A fibrewise topology describes structures where one considers a collection of spaces or continuous maps, each associated with a point from another space, organized continuously as a family. This is formalized via a continuous function p: E→B, where E is called the total space and B the base space. For each point b in B, the subset $E(b)= P^{-1}(b)$ forms what is called the fibre over b, and these fibres vary in a continuous manner as bb moves through B.

Thus, a fibrewise topological space over B is a set equipped with a topology and a projection map p that is continuous, so that the collection of all fibres (the pre-images of points in B) can be regarded as a parameterized family of spaces, with the structure depending smoothly or continuously on the base point.

Preliminaries:

This section used the references [3], [4], [5] and [6].

Definition: - Let A and B are frbrewise sets over a base space C, with projection pa: A →C and pb B → C, respectively, a map φ: A→ B is said to be fibrewise function if paoφ = pa, in other words.

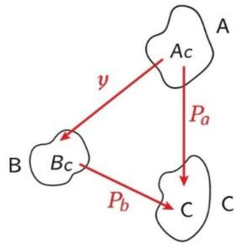

Figure 1: Fibrewise function.

if φ(Ac) ⊑ Bc for all Point c of C.

such that a fibrewise map φ: A → B over a base C Induces a fibrewise map by applying a restriction. $\varphi_{C^*}: A_{C^*} \rightarrow B_{C^*}$ over a base C* for all C* of C.

Definition: - Consider a topological space C and a fibrewise set AA over the base C. A fibrewise topological space on A is defined as any topology on A for which the projection map φ: A→C is continuous. In other words, the topology on AA must be such that the structure map φφ respects the topology of the base space C by being continuous

Definition: - Let U be a subset of a fibrewise topological space A over a base space C. Then U is called afibrewise dense subset of A over C if one of the following equivalent conditions holds:

1.For every c∈ C, the smallest fibrewise closed subset of Ac, that contains U∩ A_c is A_c itself.

2. The closure of U in A is equal to A. That is clA U= A.

3. The interior of the complement of U is empty. That is, int A (A/U) =∅.

4. Every point in Ac, where c ϵC either belongs to U.

5. for each aϵ Ac, where c ϵ C, every neighborhood Vc of a intersects U; that is Vc ∩ U ≠Ø, c ϵ C.

6. A intersects every non-empty open subset of Ac, where c ϵ C

Definition: - The fibrewise function φ: A→B, where A and B are fibrewise topological spaces over a base C, is defined as follows:

a. φ is fibrewise continuous if, for each a∈Ac with c∈C, the inverse image φ−1(U) of every open set U⊆B containing φ(a) is an open set in Ac.

b. φ is fibrewise open (respectively, fibrewise closed) if, for each a∈Ac with c∈C, the image of every open (respectively closed) set U⊆Ac containing a is an open (respectively closed) set V⊆Bc containing φ(a).

Definition: - A fibrewise topological space A over a base C is called fibrewise closed (respectively, fibrewise open) if the projection map ρ: A→C is a closed map (respectively, an open map).

Definition: - (Embedding): A function f:(A, τ) →(B,τ^*) is called an embedding if it satisfies the following properties:

- Injective (one-to-one),
- Continuous, and
- Open onto its image (i.e., it is an open map when considered as a function onto the subspace f(A)⊆B.

More precisely, an embedding is an injective continuous map that induces a homeomorphism between A and its image f(A) endowed with the subspace topology inherited from (B,τ^*)

Definition: - A map h: (A,τ) → (B,τ*) is a map homeomorphism (topological mapping) if h is one- to - one, onto, and h, h^{-1} are continuous.

Definition: - Let A^* be a fibrewise topological space over a base C and let A⊆A^* be a subspace of A∗. The fibrewise map:

$$\lambda a: A \rightarrow A^*,\ \lambda a\ (a) = a,$$

is called a fibrewise embedding if it is fibrewise continuous and, for every open set F⊆A^*, the preimage satisfies.

$$\lambda a^{-1}(F) = A \cap F,$$

where, the set A∩F is open in A.

Consequently, the subspace A is said to be fibrewise open (respectively, fibrewise closed) if and only if the fibrewise subspace A is fibrewise open (respectively, fibrewise closed) in A^*.

Definition: - Let {(Aα, τα)} α∈Λ {(Aα, τα)} α∈Λ be a family of fibrewise topological spaces over a common base space C. Then the fibrewise product.

$$A=\prod_{\alpha\in\Lambda}A_\alpha$$

is defined as the fibrewise space over C equipped with the family of fibrewise projection maps

$$\pi_\alpha: A \to A_\alpha$$

where each πα maps a point in the product to its α-th component.

$p_\alpha o\ \pi_\alpha : \prod_{\alpha\epsilon\Lambda} A_\alpha \to$ C. Where π_α: $\prod_{\alpha\epsilon\Lambda} A_\alpha \to A_\alpha$ is the α – the projection of A_α.

This definition generalizes the usual product topology to the fibrewise setting and aligns with the standard notion found in fibrewise topology literature.

Definition: - Let (A,τ) be a fibrewise topological space over a base set C. The space A is said to be a fibrewise Hausdorff (T_2) space if, for each point c in C, the fibre Ac is a Hausdorff space. More precisely, for any two distinct elements a_1,a_2∈Ac, with a1$a_1 \neq a_2$, there exist open sets w_1 and w_2 in Ac such that $a_1 \in w_1$,$a_2 \in w_2$, and $w_1 \cap w_2 = \emptyset$.

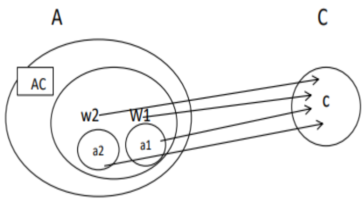

Figure 2: Fibrewise Hausdorff (T_2).

Definition: - Let (A,τ) be a fibrewise topological space over a base set C. Then (A,τ) is called ***a fibrewise*** $T_{3\frac{1}{2}}$ ***space*** if and only if it is both fibrewise completely regular and fibrewise T_1. That is, for every point c∈C, the fibre (Ac,τc) is a completely regular T_1 topological space.

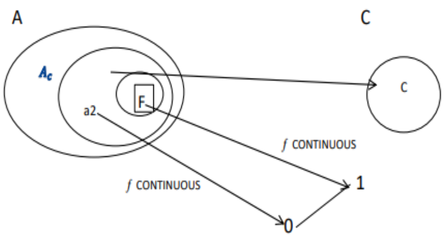

Figure 3: Fibrewise ($T_{3\frac{1}{2}}$).

Fibrewise Compact Spaces:

Definition: - A fibrewise compact space over a base C is a fibrewse space in which for each open cover of the space contains a finite subcover.

Definition: - A map α: A→B is called proper if it satisfies the following conditions:

- α is a continuous function,
- α is a closed map (i.e., the image of every closed set in A is closed in B),
- For every point b∈B, the preimage α^{-1}(b) is a compact subset of A.

Definition: - The fibrewise topological space A over a base C is called fibrewise compact if the protection P is proper***.***

Proposition: - A fibrewise topological space (A,τ) over a base space C is said to be fibrewise compact if and only if two conditions are met: first, the subset A is fibrewise closed in the total space, and second, each fibre Ac=φ^{-1}(c) for every c∈Cis a compact topological space.

This means that compactness is preserved on each fibre individually, while the entire set A remains suitably closed with respect to the fibrewise structure.

proof/ ⇒let A be a fibrewise compact space, then the projection p: A→ C is proper function i-e, p is closed and for each c∊C, Ac is compact.
Hence A is fibrewise closed and every fibre of A is compact.
⇐let A be fibrewise closed and every fibre A_c is compact, then the projection p: A→C is closed and it is clear that p is continuous, also for each c ϵ C, Ac is compact. Hence A is fibrewise compact.

Theorem: - let A and B are fibrewise topological spaces over a base C, such A be a fibrewise compact space over a base C, and h: A →B fibrewise continuous, surjection maping A onto B. Then B is fibrewise compact space over a base C.

Proof let V an open cover of B. Use the continuity of h to bull is back to an open cover of A. Use compactness to extract a finite subcover for A, and then use the fact that h is onto to reconstruct a finite subcover for B.

Gorollary: - 3.6 Let (A,τ) be a fibrewise compact space over a base C, B is fibrewise space over a base C, and h: A→ B a fibrewise continuous maping. The image h(a) of A in B is a fibrewise compact space over a base C of B.

proposition: - let (A, τ) be a compact fibrewise topological space over a base C, and let (B, δ) be a housdorff fibrewise topological space over a base C. Then any fibrewise continuous maping, bijection maping h: A→ B is a homeomorphism.

proof/ Since h: A→ B is fibrewise continuous maping, bijection maping
then h is open, or equivalently closed. So, let F ⊆ Ac, where c ∈ C be a closed set, then F is compact set in Ac, and therefore h (F) is compact, so we have that h(F) is closed. Thus, h is a homeomorphism.

Proposition:

The fibrewise topological product

$$A=\prod \beta Ar$$

is fibrewise compact if each Ar is fibrewise compact over a base C for all r∈R.

Proof: By considering the finite case, for finite products, a direct and concise argument suffices. Consequently, Given A and B be fibrewise topological spaces over a base C. If A pxidy: A ac B CAPB = B is proper. If b is fibrewise compact as well, then so is A ac B. The result extends similarly to finite coproducts.

Fibrewise Compactifications: -

This section explores and examines several new concepts:

Definition: - let (A, τ) and (B, δ) are fibrewise topological spaces over a base C, a fibrewise compact space (B, δ) is a fibrewise compactification of a fibrewise space (A, τ) iff there exist a mapping h of A into B such that h is homeomorphism of A onto the subspace h(a) of B and h(a) is fibrewise dense set in B.

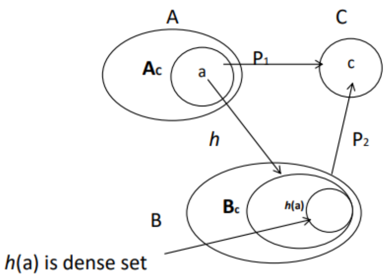

Figure 4: Fibrewise Compactification.

Definition: - Afibrewise compactification of (A, P) is atriple (B, Q,q), where (B, δ) is fibrewise compact space over a base C, q: B →C a continuous proper map and Q : A → B a homeomorphism into an open dense subset of B such that p = q o Q .

Example: - let A = R with the usual topology and C =R with the usual topology, P: A → C define by p(a) = a², then (A, τ) is fibrewise topological space over a base C, and Ac = $\begin{cases} \{-\sqrt{c}, \sqrt{c}\} & \text{if } c > 0 \\ \{o\} & \text{if } c = o \\ \emptyset & \text{if } c < o \end{cases}$

if $c > 0$, Ac = $\{-\sqrt{c}, \sqrt{c}\}$ is nontrivial subspace and the subspace is discrete subspace of A, and let B = (N, δ) ; δ = {N, ∅ . E*, C+}.

defined by identity map over a base C, then (B, δ) is fibrewise topological space over a base C, and is fibreswise compact space, and if h: (A, τ) → (B, δ) such that h is contant map, then (B, δ) is fibrewise compactification of fibrewise (A, τ) over a base C.

proposition: - let A be a fibrewise topological space over a base C. If there exists an embedding J: A → B such that B is a fibrewise compact hausdorff space over a base C, then there exists an embedding J1: A → Z such that Z is a compact hausdorff and J1 (a) = Z.

proof / If J: A → B embedding, such B is fibrewrise compact hausdorff. Then Z= $\overline{J(a) \subseteq B}$ j1: A → τ ia →j(a) ⇒ j1(a) is dense set.

Corollary: - let A be afibrewise topological space. The following conditions are equivalents:

1/ There exists a fiberwise compactification of A.

2 / There exists a fibrewise embedding J: A → B where B is a fibrewise compat hausdorff space.

proof/ Follows from proposition (3-4)

Theorem: - 4.6 A fibrewise space A has a compactification iff A is fibrewise regular (i-e) is a T3$\frac{1}{2}$ space.

proof/ ⇒ let f: A → B compactification. of A, B is fibrewise compact hausdorff, then B is normal (T4), B is T3$\frac{1}{2}$, then f(a) ⊆ B is T3$\frac{1}{2}$, f(a) ≅ A is T3$\frac{1}{2}$.

⇐ 1) {f: A → [0, 1]} separated points from closed sets, F⊆A (closed), a ϵ A/F, there is f: A → [0, 1] such F→0 , a→f(a) >0

2) such family defines embedding J: A → $\pi_{i\epsilon I}$ [0,1], a → (f (a)) $_{i\epsilon z}$

Note: let K(a) is family of all continuous functions. f: A → [0, 1], A is completely regular, then K(a) separated points from closed sets, then there is an embedding Ja: A → $\pi_{f\epsilon k(x)}$ [0,1] is compact hausdarff,

aϵ(f(a)) $_{f\epsilon k(x)}$, Up shat; we have a compactification.

Definition: - Suppose (B_2, p_2), (B_1, p_1), one fibrewise compactification of fibrewise A over a base C. We say that (B_2, p_2) ≥ (B_1, p_1) if there exists fibrewise continuous function f: B_2 → B_1 such that fop_2 = p_1 where A_1, B_1, and B_2 are fibrewise spaces over a base C.

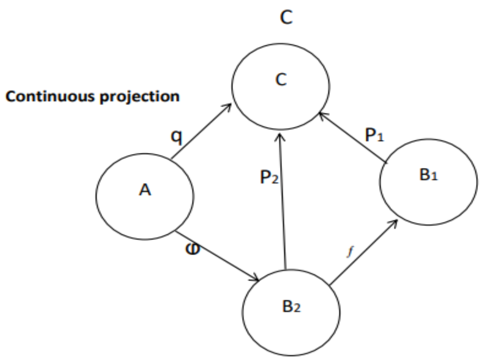

Figure 5: (B_2, p_2) ≥ (B_1, p_1).

The Stone-Cech Compactification:

This section discusses some concepts and properties of the largest compactification of fibrewise topological space over a base C.

Definition: - let A be any fibrewise topological space over a base C, then the Largest compactification of A is called the stone cech- compactification and is denoted by βA.

In other words, for a fibrewise Tychnoff space A over a base C, there is a unique (up to equivalence) compactification that in some sense all Continuous maps from A into all fibrewise compact hausdorff Space. This is called ***the stone cech- compactification***.

Theorem: - If every Tychonoff fibrewise space A has of ***a largest compactification βA***, then any product of fibrewise compact housdorff spores over a base C is fibrewise compact space over a base C.

proof/ Suppose $\{A_\alpha\}_{\alpha\epsilon\wedge}$is a family of fibrewise compact T2- space over a base C. Since $\pi_{\alpha\epsilon\wedge}\ A_\alpha$ is Tychonoff and it has a compactification β ($\pi_{\alpha\epsilon\wedge}\ A_\alpha$), such α the projcation map π_α can be extended to π_π^β: $\beta(\pi_{\alpha\epsilon\wedge}\ A_\alpha)$ → A_α

for a ϵ $\beta(\pi_{\alpha\epsilon\wedge}\ A_\alpha)$ apoint f (a) ϵ A with coordinates

f(a)(α) = π_π^β ; f : ($\pi_{\alpha\epsilon\wedge}\ A_\alpha$) → $\pi_{\alpha\epsilon\wedge}\ A_\alpha$

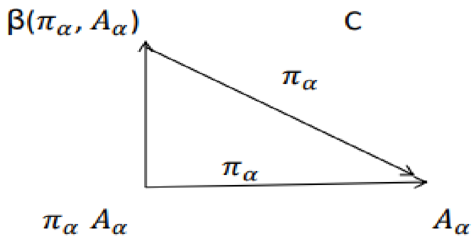

Figure 6: Fibrewise Product.

then f is a continuous. If a ϵ $\pi_{\alpha\epsilon\wedge}$ $A_\alpha \subseteq \beta$ ($\pi_{\alpha\epsilon\wedge}$ A_α) then f(a)(α) = π_π^β(a) = π_α(a) = a(α) = a_α for each α , so f(a) = a . Therefore $\pi_{\alpha\epsilon\wedge}$ A_α is continuous image of the compact space $\beta(\pi_{\alpha\epsilon\wedge}$ $A_\alpha)$. Thus $\pi_{\alpha\epsilon\wedge}$ A_α is fibrewise compact over a base C.

Theorem: Suppose B_1, and B_2 one fibrewise compactification of A over a base C, where the embeddings are the identity maps. Then (B_1, i) ≅ (B_2, i) if for every pair of disjoint closed sets in A, clb_1 U ∩ clb_1 V= ∅ ⟺ cl b_2 U ∩ clb_2 V= ∅ .

Proof/ ⇒ If (B_1, i) ≅ (B_2, i), it is clear that clb_1 U ∩ clb_1 V = ∅ ⟺ cl b_2 U ∩ clb_2 V = ∅.
⇐ If clb_1 U ∩ clb_1 V = ∅ ⟺ cl b_2 U ∩ clb_2 V = ∅ then the identity maps i_1: A→ B_1 and i_2: A→ B_2 can be extended to maps f_1: B_2→ B_1 and f_2: B_1 → B_2. That is clear f_1o f_2/a and f_2o f_1/a are identity maps on the dense subspace A. Therefore, f_1of_2 and f_2of_1 are each the identity everywhere. Thus f_1, f_2 are homoemorphism. So (B_1, i) ≅ (B_2, i).

Lemma: - If π: A → B is a fibrewise surjection between fibrewise Tychonoff spaces A and B, the map $\beta\pi$: βA → βB is also surjective.

proof/ Consider a surjection of Tychonoff spaces, π: A→ B from the compactness of βA, we have the image of βπ compact, hence closed. But the image of βπ continuous β, a dense subset of βb; so, the image must cancide with βb.

Remark: - If A is a fibrewise Tychonoff space over a base C, then the map from A to its image in βA is a homeomorphism, so A can be thought of as a (dense) subspace of βA.

proposition: - let {($A_\propto$, p_a): a ϵ ∧} be a collection of fibrewise spaces.
1/ If each p_a is T_i (i = 0,1,2) then the product p is also Ti (i,0,1,2)
2/ If each p_a is a surjective T3$\frac{1}{2}$ map, then the product p is also a T3$\frac{1}{2}$ map.
3/ If each $P_\propto$ is a compact map, the product p is a compact map.
From above Remark and proposition we write the following results.

Results: - let $a_\propto$, be a fibrewise compact hausdorff space over a base C for each ∝ ∈∧ , then
1/ If $A_\propto$ be a fibrewise Tychonoff space over a base C for each ∝ ∈∧ , then $\pi A_\propto$, is a ston Cech-compactification βA of $A_\propto$ with fibrewise continuous map, and so the map is homeomorphism
2/ $\pi A_\propto$ is *stone Cech -compactification βA*, with any fibrewise continuous function.

Conclusion

In this paper, we have explored the concept of fibrewise topological spaces, their compactness, and compactifications, emphasizing the Stone-Čech compactification as the largest fibrewise compactification. Our results demonstrate that every fibrewise continuous function extends uniquely to the Stone-Čech compactification, highlighting its significance in the theory. These findings contribute to a deeper understanding of fibrewise topology and its applications.

References:

[1] B. Guo and Y. han.‘‘ a brief introduction to fibrewise topological spaces theory’’. *journal of mathematical research with applications*, vol. 32, no.5, pp.626-630, sept. 2012.
[2] I. M. James,‘‘ Book reviews- fibrewis topology’’. *Bulletin of the American mathematical society*, vol.24, no.1, pp.246-248, Jan.1991
[3]S.S.Mohamoud and y.y.yousif.‘‘ fibrewise near separation axioms’’. *international mathematical forum*, vol.7, no.35, pp.1725-1736, 2021
[4] M.E.Abd El-monsef, A.E.Radwan, and Y.Y.Yousef. "Fibrewise near compact and locally near compact spaces", *Mathematical Theory and Modeling,* vol. 3, no.8, 2013.
[5] N.M.Alsharrani, "Separation axioms in fibrewise ideal topological spaces", M.A.thesis, Alasmarya Islamic University, Libya, 2022.
[6] M. Clara, U. Neira, " Akuratowski-Mrowka type characterization of fibrewise compactness", *Applied general topology,* vol.12, no. 1, pp 41- 47, 2011.