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The Convergence of The New Preconditioned SOR Iterative Method for Solving The Linear System

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تقارب التقنيات التكرارية الجديدة المشروطة لطريقة الاسترخاء لحل النظام الخطي

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Abstract:
The SOR is a basic iterative method for solution of the linear system$Ax = b$. Such systems can easily be solved using direct methods such as Gaussian elimination. However, when the coefficient matrix A is large and sparse iterative methods such as the SOR become indispensable. New preconditioned for speeding up the convergence of the SOR iterative method for solving the linear system $Ax = b$ is proposed. Arising from the preconditioned Two forms of the new preconditioned iterative techniques of the SOR method are developed. The preconditioned iterations are applied to the linear system whose coefficient matrix is an M-matrix. Convergence of the preconditioned iterations is established through standard procedures. Numerical examples and results comparison are conformity with the analytic results. More so, it is established that the spectral radii of the proposed preconditioned T_{Psor} *and* T_{Psor2} are less than that of the classical SOR, which implies faster convergences.

Keywords: Convergence, SOR Method, Preconditioned, M-matrix, Iterative Matrix, Spectral Radius.

الملخص:
طريقة الاسترخاء هي طريقة تكرارية أساسية لحل النظام الخطي، حيث يمكن حل هذه الأنظمة بسهولة باستخدام طرق مباشرة مثل الحذف الغاوسي. ومع ذلك، عندما تكون مصفوفة المعاملات كبيرة ومتفرقة تصبح الطرق التكرارية مثل طريقة الاسترخاء ضرورية. تم اقتراح طريقة جديدة مسبقة لتسريع تقارب طريقة الاسترخاء التكرارية لحل النظام الخطي، بناء على هذه الطريقة المسبقة تم تطوير شكلين من التقنيات التكرارية الجديدة مسبقة التكييف لطريقة الاسترخاء، يتم تطبيق التكرارات المسبقة التكييف على النظام الخطي الذي تكون مصفوفة معاملاته عبارة عن مصفوفة (م) يتم تحديد تقارب التكرارات المسبقة التكييف من خلال إجراءات قياسية. تتوافق الأمثلة العددية ومقارنة النتائج مع النتائج التحليلية. علاوة على ذلك، ثبت أن أنصاف الأقطار الطيفية للتقنيات التكرارية الجديدة المسبقة التكييف المقترحة أقل من تلك الخاصة بالطريقة الكلاسيكية، مما يعني تقارب أسرع.

الكلمات المفتاحية: تقارب، طريقة الاسترخاء، مشروط مسبقاً، مصفوفة-م، مصفوفة تكرارية، نصف القطر الطيفي.

Introduction

Finding solution of the linear system:

$$Ax = b \quad (1)$$

where $A = [a_{ij}]$ is an $n \times n$ real nonsingular matrix of known values called the coefficient matrix of the system, $b = [b_{ij}]$ is an $n \times 1$ real vector of known values and $x = [x_i]$ is the $n \times 1$ vector of unknowns, let $\mathcal{M}_n$ be the set of all $n \times n$ matrices, $\mathbb{R}^n$ be the set of all $n \times 1$ real vectors and let $x^* \in \mathbb{R}^n$ denote the unique solution of (1).

In constructing Stationary Iterative Methods for the solution of the Linear System (1). The basic idea is to split the coefficient matrix $A \in \mathcal{M}_n$ as following.

$$A = M - N \tag{2}$$

where M, N are nonsingular matrices. So, the linear system Ax = b is equivalent to the linear system $(M - N)x = b$ or $Mx = Nx + b$ or

$$x = M^{-1}Nx + M^{-1}b \tag{3}$$

Then, given $x^{(0)} \in \mathbb{R}^n$ equation (3) suggests the iterative method

$$x^{(k+1)} = M^{-1}Nx^k + M^{-1}b \qquad k = 0, 1, 2, \ldots.. \tag{4}$$

where, $T = M^{-1}N$ is the iteration matrix and $c = M^{-1}b$ Different choices of M and N lead to different stationary iterative methods.

Remark: For the coefficient matrix $A \in \mathcal{M}_n$ of the linear system $Ax = b$, author consider splitting of the form $A = D - L - U$. Under the assumption that D is nonsingular, author has given Jacobi, Gauss-Seidel and SOR iteration matrices T_J , T_G and, T_{SOR} by

$T_J = D^{-1}(L + U)$, $T_G = (D - L)^{-1}U$ and $T_{SOR} = (D - \omega L)^{-1}[(1 - \omega)D + \omega U]$. Since D is nonsingular, without loss of generality.

known as the Successive Over Relaxation (SOR) Method. It is clear that it can be performed whenever D is nonsingular. Notice that $\omega = 1$ gives the Gauss-Seidel method.

Theorem 1([1] pp457): For any $x^{(0)} \in \mathbb{R}^n$, the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by:

$$x^{(k)} = Tx^{(k-1)} + c \quad , \qquad k = 0, 1, 2, \ldots.. \tag{5}$$

Converges to the unique solution of $x = Tx + c$ if and only if the spectral radius $\rho(T) < 1$.

So, $\rho(T)$ being the spectral radius of the iteration matrix of a convergent stationary iterative method for $Ax = b$. Author has seen that the Jacobi, Gauss-Seidel and SOR iterative techniques can be written as,

$$x^{(k)} = T_J x^{(k-1)} + c_j \,, \quad x^{(k)} = T_G x^{(k-1)} + c_g$$
$$x^{(k)} = T_{SOR} x^{(k-1)} + c_{sor}$$

using the matrices

$$T_J = D^{-1}(L + U) \;\; \text{and} \;\; T_G = (D - L)^{-1}U$$
$$T_{SOR} = (D - \omega L)^{-1}[(1 - \omega)D + \omega U]$$

If $\rho(T_G) < 1 \; or \; \rho(T_J) < 1$, $\rho(T_{SOR}) < 1$ then the corresponding sequence $\{x^{(k)}\}_{k=0}^{\infty}$ will converge to the solution x of $Ax = b$. For example, the Jacobi scheme has

$$x^{(k)} = D^{-1}(L + U) \;\; x^{(k-1)} + D^{-1}b$$

and, if $\{x^{(k)}\}_{k=0}^{\infty}$ Converges to x, then

$$x = D^{-1}(L + U) \;\; x + D^{-1}b$$

This implies that

$$Dx = (L + U)x + b \quad and \quad (D - L - U)x = b$$

Since $D - L - U = A$, the solution x satisfies $Ax = b$:

First, author introduces some classes of matrices for which the preconditioning is applied

Definition 1 ([2] pp542): An $n \times n$ matrix $A = [a_{ij}]$ is called strictly diagonally dominant (SCDD), if for all $i = 1, 2, \ldots..n$,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \tag{6}$$

Definition 2 ([3] M. Usui): Let $A = [a_{ij}]$ be an $n \times n$ matrix. The matrix A is a Z-matrix if $a_{ij} \leq 0$, $i, j = 1, 2, \ldots..n, i \neq j$. it is an L-matrix if $a_{ii} > 0$; $i = 1, 2, \ldots..n$, and $a_{ij} \leq 0$ for all $i, j = 1, 2, \ldots..n, i \neq j$, and it is an M-matrix if A has the form $A = rI - K$, where $K \geq 0$ and $r \geq \rho(K)$.

Definition 3 ([6] A. Hadjidimos): For any A is a matrix in $\mathbb{R}^{n \times n}$ be an M-matrix if and only if $a_{ij} \leq 0$, $i \neq j = 1, 2, \ldots..n$, then A is nonsingular and $A^{-1} \geq 0$.

New Preconditioned Linear Systems

The importance of the spectral radius, in case of convergence is clear from the last. Theorem (1). Roughly speaking, the smaller the spectral radius, the faster the convergence of the iterative method. Author can transform it to the equivalent preconditioned linear system:

$$PAx = Pb$$

Or $\tilde{A}x = \tilde{b}$ where the preconditioned $P \in \mathcal{M}_n$ is nonsingular, $\tilde{A} = PA$ and $\tilde{b} = Pb$. So, author need to construct the preconditioned P in such a way that $\rho(\tilde{T}) < \rho(T) < 1$, and hence, to achieve a faster convergence to the unique solution x^*, or equivalently, a reduction in the number of iterations. Author proposes a new preconditioned P of the form:

$$P = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ w & w & \dots & w & 1 & w & \dots & w & w \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad (7)$$

where w is the parameter that will be determined and it is taking place at the *m*-th row. Notice that due to the structure of the preconditioning matrix P, the coefficient matrix A of the linear system $Ax = b$ and the coefficient matrix PA of the preconditioned linear system $PAx = Pb$ differs only at the *m*-th row. For that reason, in the forthcoming discussions author shall consider only the *m*-th rows of the coefficient matrices A and PA.

Note that the preconditioned matrix $P = (I + Pw)$ is more general than that in (7). Than

$$(I + Pw)Ax = Pb$$
$$\tilde{A} = (I + Pw)(I - L - U)$$
$$= I - L - U + P_w - P_wL - P_wU$$
$$= I - L - U - L_P - U_P + D_1 - L_1 - U_1$$

Where $P_w = -L_P - U_P$ and $-P_wL - P_wU = D_1 - L_1 - U_1$, Therefore:

$$\tilde{A} = (I + D_1) - (L + L_P + L_1) - (U + U_P + U_1)$$

or equivalently

$$\tilde{A} = \widetilde{D} - \tilde{L} - \widetilde{U}$$

Where $\widetilde{D} = (I + D_1)$ is the diagonal, $\tilde{L} = (L + L_P + L_1)$ and $\widetilde{U} = (U + U_P + U_1)$ are strictly lower and strictly upper triangular parts of $\tilde{A}$ respectively. The application of Over Relaxation Parameter ω to the preconditioned linear system (7) author obtained:

$$\omega\tilde{A}x = \omega\tilde{b}$$

Where:

$$\omega\tilde{A} = \omega(\widetilde{D} - \tilde{L} - \widetilde{U})$$
$$= \omega\widetilde{D} - \omega\tilde{L} - \omega\widetilde{U}$$
$$= \widetilde{D} - \widetilde{D} + \omega\widetilde{D} - \omega\tilde{L} - \omega\widetilde{U}$$
$$= (\widetilde{D} - \omega\tilde{L}) - (1 - \omega)\widetilde{D} - \omega\widetilde{U}$$
$$\omega\tilde{A} = (\widetilde{D} - \omega\tilde{L}) - [(1 - \omega)\widetilde{D} + \omega\widetilde{U}]$$

The similarly type1, the splitting for the preconditioned coefficient matrix $\omega\tilde{A} = M - N$

Where $M = (\widetilde{D} - \omega\tilde{L})$ and $N = (1 - \omega)\widetilde{D} + \omega\widetilde{U}$. Hence, the following preconditioned SOR iterative scheme is defined as:

$$\omega\tilde{A}x = \omega\tilde{b}$$
$$\left[(\widetilde{D} - \omega\tilde{L}) - [(1 - \omega)\widetilde{D} + \omega\widetilde{U}]\right]x = \omega\tilde{b}$$
$$(M - N)x = \omega\tilde{b}$$
$$x = M^{-1}Nx + M^{-1}(\omega\tilde{b})$$

Author can say

$$x^{(k+1)} = T_{Psor}x^{(k)} + c \quad (8)$$

Where $T_{Psor} = M^{-1}N = (\widetilde{D} - \omega\tilde{L})^{-1}[(1 - \omega)\widetilde{D} + \omega\widetilde{U}]$, be the **Type-1** for the new preconditioned SOR iteration matrix and $c = M^{-1}(\omega\tilde{b})$. Different choices of M and N lead to different stationary iterative methods and $0 < \omega < 1$. Another way author can apply of Over Relaxation Parameter ω to the preconditioned linear system (7) with change in the splitting of $\omega\tilde{A} = M - N$.

Where:

$$\omega\tilde{A} = \omega(\widetilde{D} - \tilde{L} - \widetilde{U})$$

$= \omega(I - D_1) - \omega\tilde{L} - \omega\widetilde{U}$

$$= I - \omega\tilde{L} - I + \omega(I + D_1) - \omega\widetilde{U}$$

$$= (I - \omega\tilde{L}) - I + \omega I + \omega D_1 - \omega\tilde{U}$$
$$= (I - \omega\tilde{L}) - (1-\omega)I - \omega(\tilde{U} - D_1)$$

Thus

$$\omega\tilde{A} = (I - \omega\tilde{L}) - [(1-\omega)I + \omega(\tilde{U} - D_1)]$$

the splitting for the preconditioned coefficient matrix $\omega\tilde{A} = M - N$ Where $M = (I - \omega\tilde{L})$ and $N = [(1-\omega)I + \omega(\tilde{U} - D_1)]$. Hence, the following preconditioned SOR iterative scheme is defined as:

$$\omega\tilde{A}x = \omega\tilde{b}$$
$$\left[(I - \omega\tilde{L}) - [(1-\omega)I + \omega(\tilde{U} - D_1)]\right]x = \omega\tilde{b}$$
$$(M - N)x = \omega\tilde{b}$$
$$x = M^{-1}Nx + M^{-1}(\omega\tilde{b})$$

Author obtained:

$$x^{(k+1)} = T_{Psor2}x^{(k)} + c \tag{9}$$

where,$T_{Psor2} = M^{-1}N = (I - \omega\tilde{L})^{-1}[(1-\omega)I + \omega(\tilde{U} - D_1)]$ The Type-2 for the new preconditioned SOR iteration matrix and $c = M^{-1}(\omega\tilde{b})$. Different choices of M and N lead to different stationary iterative methods and $0 < \omega < 1$.

Convergence Analysis

Convergence of the new preconditioned iterative schemes (8) (9) are established by showing that the spectral radius of the T_{Psor} and T_{Psor2} are less than 1 in each case.

Lemma 1 ([4] varga): For any A is an irreducible $n \times n$ matrix such that $A \geq 0$,then

1- A has a positive real eigenvalue that is equal to its spectral radius.
2- There exists a corresponding eigenvector $x > 0$, for the eigenvalue
3- The spectral radius$\rho(A)$ increases if any entry of A increases.
4-The spectral radius $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 ([5] Li and sun): Let $T = M^{-1}N$ be an M, N are splitting of A,Then the splitting is convergent, that mean $\rho(M^{-1}N) < 1$, if and only if A is a nonsingular M-matrix.

Theorem 2 ([9] pp290): Let A is symmetric positive definite, then $\rho(T_{Psor}) < 1$ for all relaxation parameters in the range $0 < \omega < 1$,so SOR converges for all $0 < \omega < 1$. when author consider $\omega = 1$ it becomes evident that the Gauss-Seidel method similarly exhibited converges.

Theorem 3 ([8] pp150): Let $\omega \in \mathbb{R}$, author has

$$\rho(T_{Psor}) < |1 - \omega|$$

Proof: when taking the determinant of matrix T_{Psor} author gotten:

$$det(T_{Psor}) = det\left[(\tilde{D} - \omega\tilde{L})^{-1}[(1-\omega)\tilde{D} + \omega\tilde{U}]\right]$$
$$= det\left[(\tilde{D} - \omega\tilde{L})^{-1}\right].det[(1-\omega)\tilde{D} + \omega\tilde{U}]$$

Since the matrices here are triangular, their determinants are equal to the product of their diagonal entries, so author has $det(T_{Psor}) = (1-\omega)^n$. The determinant of T_{Psor} is also equal to the product of its eigenvalues, and it follows that at least 1 of the n eigenvalues must have absolute value greater than or equal to$|1 - \omega|$.

Theorem 4 ([7] A. Ndanuse): Let $T_{SOR} = (I - \omega L)^{-1}[(1-\omega)I + \omega U]$ be the SOR iteration matrix while $T_{Psor} = (\tilde{D} - \omega\tilde{L})^{-1}[(1-\omega)\tilde{D} + \omega\tilde{U}]$ and $T_{Psor2} = (I - \omega\tilde{L})^{-1}[(1-\omega)I + \omega(\tilde{U} - D_1)]$ are the the preconditioned SOR iteration matrices. if A is an irreducible M-matrix with the conditions $0 \leq a_{1,i}a_{i,1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 2, 3, \ldots.. n$ and $0 < \omega < 1$ then T_{SOR} ,T_{Psor} and T_{Psor2} are irreducible and nonnegative matrices.

Proof: From A is an M-matrix, $L \geq 0$ and $U \geq 0$. Then,

$$(I - \omega L)^{-1} = I + \omega L + \omega^2 L^2 \ldots \ldots \ldots + \omega^{n-1}L^{n-1} \geq 0$$
$$(1-\omega)I + \omega U \geq 0$$

For $0 < \omega < 1$, Thus $T_{SOR} = (I - \omega L)^{-1}[(1-\omega)I + \omega U] \geq 0$ Hence, T_{SOR} is a nonnegative matrix.

Corollary 1 ([7] A. Ndanuse): Let $T_G = (1 - L)^{-1}U$ be the Gauss-Seidel iteration matrix and $T_{PG} = (\tilde{D} - \tilde{L})^{-1}\tilde{U}$ be the preconditioned Gauss-Seidel iteration matrix. If A is an irreducible M-matrix with $0 \leq a_{1,i}a_{i,1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 2, 3, \ldots.. n$ Than

1- $\rho(T_{PG}) < \rho(T_G)$, if $\rho(T_G) < 1$.
2- $\rho(T_{PG}) = \rho(T_G)$, if $\rho(T_G) = 1$
3- $\rho(T_{PG}) > \rho(T_G)$, if $\rho(T_G) > 1$

Numerical Results

In This part. author compares preconditioned SOR methods for the following the problems.

Example 1: let us consider the matrix A of the form

$$A = \begin{bmatrix} 1 & -0.1 & -0.2 & -0.3 \\ -0.4 & 1 & -0.5 & -0.1 \\ -0.3 & -0.2 & 1 & -0.1 \\ -0.2 & -0.1 & -0.3 & 1 \end{bmatrix}$$

In this Example: author uses new preconditioning of SCDD, M-matrix whit change rows in every case and compares the spectral radiuses for T_G, T_J and T_{Psor} Iterative matrices. The results are given in *Table1* for preconditioning Iterative matrices (single row and all rows) T_G are gives better results, Than the new preconditioned SOR methods

Table 1: Result of the spectral radiuses of T_G, T_J and T_{Psor}.

row	T_G	T_J	T_{Psor}
w_1	0.32356	0.59496	0.94323
w_2	0.38801	0.63334	0.96214
w_3	0.31356	0.5744	0.95608
w_4	0.3782	0.62604	0.96136
All w	0.1412	0.42441	0.94121

Example 2: the same coefficient matrix A in Example 1.

In this Example author compares the spectral radiuses for, T_{sor}, T_{Psor} and T_{Psor2}r for some values of relaxation parameter $0 < \omega < 1$. It reveals that the T_{Psor} ,$T_{Psor2}$are faster convergence than T_{sor} The results were as follows in Table 2.

Table 2: Result of the spectral radiuses of T_{sor}, T_{Psor} and T_{Psor2}

ω	T_{sor}	T_{Psor}	T_{Psor2}
0.1	0.96562	0.94293	0.94124
0.3	0.88818	0.81289	0.81288
0.5	0.79590	0.66802	0.66802
0.7	0.68199	0.49918	0.49919
0.9	0.53202	0.28804	0.28804

Corollary 2: Let $T_{sor} = (I - \omega L)^{-1}[(1-\omega)I + \omega U]$ be the SOR iteration matrix and $T_{Psor2} = \left(I - \omega\tilde{L}\right)^{-1}\left[(1-\omega)I + \omega\left(\tilde{U} - D_1\right)\right]$ be the preconditioned SOR iteration matrix. If A is an irreducible M-matrix with $0 \le a_{1,i}a_{i,1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 2, 3, \dots\dots n$ Than

1- $\rho(T_{Psor2}) < \rho(T_{sor}),$ if $\rho(T_{sor}) < 1.$

2- $\rho(T_{Psor2}) = \rho(T_{sor}),$ if $\rho(T_{sor}) = 1.$

3- $\rho(T_{Psor2}) > \rho(T_{sor}),$ if $\rho(T_{sor}) > 1.$

Conclusions

In this work, author has introduced new preconditioned for SOR method for the solution of Linear System $Ax = b$. Author used two different forms of the new preconditioned T_{Psor} ,T_{Psor2} Types iterations are for the preconditioned. Author has proven that the spectral radius of the iteration matrix of preconditioned T_{Psor} method is smaller than that of the classical T_{sor} method, if the relaxation parameter $0 < \omega < 1$. author concludes that the new preconditioned SOR methods is faster convergence properties than the classical SOR.

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