The North African Journal of Scientific Publishing (NAJSP)
مجلة شمال إفريقيا للنشر العلمي (NAJSP)
E-ISSN: 2959-4820
Volume 3, Issue 2, April - June 2025
Page No: 92-98
Website: https://najsp.com/index.php/home/index
SJIFactor 2024: 5.49 | معامل التأثير العربي (AIF) 2024: 0.71 | ISI 2024: 0.696

The q-Integral Operator of Meromorphic Functions

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المؤثر التكاملي- q للدوال الميرومورفية

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Received: February 03, 2025	Accepted: April 06, 2025	Published: April 21, 2025

Abstract:
In 2012, Frasin presented sufficient conditions for the integral operator $H(z)$ associated with meromorphic functions. This study aims to extend Frasin's results by utilizing q-calculus to introduce and examine the q- integral operator $\mathcal{H}_q(z)$ by using q-hypergeometric function. The research focuses on deriving sufficient conditions for this operator and studying its properties.

Keywords: The q-hypergeometric function, q-derivative, q-Jackson integral, Meromorphic functions.

الملخص
في عام 2012، قدم فراسين شروطا كافية للمؤثر التكاملي $H(z)$ المرتبط بالدوال الميرومورفية. تهدف هذه الدراسة إلى توسيع نتائج فراسين من خلال استخدام حساب التفاضل والتكامل - q لتقديم ودراسة المؤثر - q التكاملي $\mathcal{H}_q(z)$ باستخدام الدالة فوق الهندسية من النوع- q . يركز البحث على اشتقاق الشروط الكافية لهذا المؤثر ودراسة خصائصه.

الكلمات المفتاحية: الدالة فوق الهندسية من النوع - q، المشتقة -q، التكامل- q لجاكسون، الدوال الميرومورفية.

Introduction

The q- calculus is a branch of math that connects different areas like hypergeometric series, complex analysis, and particle physics. It's a popular field today with various branches, such as quantum calculus and continued fractions. In 1910, Jackson [1] introduced the definite q-integral and systematically developed q-calculus. Later in the 20th century, the field expanded as a result of its mathematical and physical applications. Several researchers studied q-integrals [2-4]. This paper introduces a new q-integral operator $\mathcal{H}_q(z)$ based on the q-hypergeometric function. It also defines conditions for this operator to belong to the class $\Omega_{N,q}(\eta)$. Before that, we review some basic ideas of q-calculus. All of the results can be found in [5-8] .

Let q be a complex number such that $q < 1$. For $n \in \mathbb{N}$, and any complex number α the q-Pochhammer symbol, also known as the q-shifted factorial is defined as follows.

$$(\alpha;q)_0 = 1, \quad (\alpha;q)_n = \prod_{k=0}^{n-1} (1-\alpha q^k). \tag{1.1}$$

The q-Pochhammer symbol can be extended to an infinite product:

$$\lim_{n\to\infty} (\alpha;q)_n = (\alpha;q)_\infty = \prod_{m=0}^{\infty} (1-\alpha q^m), \tag{1.2}$$

and

$$(\alpha; q)_n = \frac{(\alpha;q)_\infty}{(\alpha q^n;q)_\infty}, \quad n \in \mathbb{N}_0, |q| < 1. \tag{1.3}$$

The q-derivative of a function f is given by:

$$D_q f(z) = \frac{f(zq) - f(z)}{(q-1)z}, \quad q \neq 1, \ z \neq 0. \tag{1.4}$$

If $f'(0)$ exists, the q-derivative satisfies $D_q f(0) = f'(0)$. As q approaches 1, the q-derivative converges to the standard derivative. The q-Jackson integrals defined as given in [1] :

$$\int_0^z f(t) d_q t = (1-q) z \sum_{n=0}^{\infty} q^n f(zq^n). \tag{1.5}$$

For any function f, one has

$$D_q\left(\int_0^z f(t) d_q t\right) = f(z) \tag{1.6}$$

Therefor, the q-derivative of $f(z) = z^k$, is given by

$$D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}, \tag{1.7}$$

where

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q} & , \ q \neq 1 \\ k & , q = 1 \end{cases}. \tag{1.8}$$

The q-derivative D_q for the product and quotient of two functions can expressed as follows:

$$D_q(f(z)g(z)) = f(qz) D_q g(z) + g(z) D_q f(z), \tag{1.9}$$

$$D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z) D_q f(z) - f(z) D_q g(z)}{g(qz) g(z)}, \quad g(qz)g(z) \neq 0. \tag{1.10}$$

Note that:

$$D_q(\log f(z)) = \frac{\ln q}{q-1} \frac{D_q f(z)}{f(z)}. \tag{1.11}$$

Let Ω represent the set of functions expressed in the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.12}$$

analytic within the punctured open unit disk $\mathbb{U}^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = \mathbb{U} \backslash \{0\}$, where $\mathbb{U}$ denotes the open unit disk.

A function $f \in \Omega$ is considered meromorphic starlike of order α for some $(0 \leq \alpha < 1)$ if,

$$-\Re\mathrm{e}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U}^*), \tag{1.13}$$

we denote $\Omega^*(\alpha)$ as the class that includes all meromorphic starlike functions of order α.
In addition, for $f \in \Omega$, Wang et al. [9] introduced and investigated the subclass $\Omega_N(\eta)$ of Ω, which consists of functions $f(z)$ that satisfy:

$$-\Re\mathrm{e}\left(\frac{zf''(z)}{f'(z)} + 1\right) < \eta, \quad (\eta > 1, \ z \in \mathbb{U}^*). \tag{1.14}$$

Using q-derivative, we define the q-analogues of the function classes $\Omega^*(\alpha), \Omega_N(\eta)$.

A function $f \in \Omega$ belongs to the class $\Omega_q^*(\alpha)$ if it satisfies:

$$-\Re\mathrm{e}\left(\frac{zD_q f(z)}{f(z)}\right) > \alpha, \quad (0 \leq \alpha < 1, z \in \mathbb{U}^*). \tag{1.15}$$

A function $f \in \Omega$ belongs to the class $\Omega_{N,q}(\eta)$ if it satisfies:

$$-\Re\mathrm{e}\left(\frac{zqD_q^2 f(z)}{D_q f(z)} + 1\right) < \eta \quad , (\eta > \frac{1}{q}, z \in \mathbb{U}^*). \tag{1.16}$$

Given complex numbers $a_1, \ldots, a_r, b_1, \ldots, b_s$ with the condition that $b_j \neq 0, -1, \ldots$, for $j \in \{1, 2, \ldots, s\}$ the basic hypergeometric function (or the general q-hypergeometric function series) ${}_r\Phi_s$ is defined as follows:

$$ {}_r\Phi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z)=\sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}\frac{z^k}{(q;q)_k}\left[(-1)^k q^{\frac{k(k-1)}{2}}\right]^{s-r+1}, \tag{1.17} $$

where $(a_1,\dots,a_r;q)_k$ is abbreviation for $\prod_{j=1}^{r}(a_j;q)_k$, and $q\neq 0$ when $r>s+1,(r,s\in\mathbb{N}_0=\{0,1,2,\dots\})$, we observe that as $q\to 1$ we have $(1-q)^{-k}(q^{\alpha};q)_k\to(a)_k$, where $(a)_k$ known as the Pochhammer symbol , which is defined as: $(a)_k=a(a+1)\dots(a+k-1)$.

In 1846, Heine extended the concept of the hypergeometric series by introducing the q-hypergeometric series. For a short summary, see references [10], [6], and [11].

For $z\in\mathbb{U}$, $|q|<1$, and $r=s+1$, the q-hypergeometric function reduces to a special case with the following form.

$$ {}_r\Psi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z)=\sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k(q;q)_k}z^k. $$

Corresponding to the function $\mathcal{W}_{r,s}[a_1,b_1;q]$ given by:

$$ \mathcal{W}_{r,s}[a_1,b_1;q]=\frac{1}{z}\ {}_r\Psi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z), \tag{1.18} $$

We introduce the linear operator $\mathcal{Q}_{r,s}[a_1;q]:\Omega\to\Omega$ defined as follows:

$$ \mathcal{Q}_{r,s}[a_1;q]=\mathcal{W}_{r,s}[a_1,b_1;q]*f(z), $$

$$ \mathcal{Q}_{r,s}[a_1;q]f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{(a_1,\dots,a_r;q)_{k+1}}{(b_1,\dots,b_s;q)_{k+1}(q;q)_{k+1}}a_k z^k. \tag{1.19} $$

The linear operator $\mathcal{Q}_{r,s}[a_1;q]$ generalizes many previous operators. Specifically, when $\alpha_i=q^{\alpha_i},\beta_j=q^{\beta_j},\alpha_i,\beta_j\in\mathbb{C},\beta_j\neq 0,-1,\dots,$ for $(i=1,2\dots,r)$ and $(j=1,2,\dots,s),q\to 1$and $r=s+1$, we have the following:

$$ \mathcal{Q}_{r,s}[q^{a_1};q]f(z)=\mathcal{H}_r^s[a_1], $$

The linear operator $\mathcal{H}_r^s[a_1]$ was previously studied by Liu and Srivastava [12] in the case $p=1$, Furthermore, for $r=2,s=1$ and $a_2=1$, we obtain:

$$ \mathcal{Q}_{2,1}[q^a,q,q^c;q]=\mathcal{L}(a,c), $$

The linear operator $\mathcal{L}(a,c)$ was previously studied by Liu and Srivastava [13] in the case $p=1$, this operator is related to the one studied by Carlson and Shaffer, which has been widely used in analytic and univalent function theory in $\mathbb{U}$.

Many researchers have studied an integral operator involving q-hypergeometric functions[14],[15]:

$$ \frac{1}{z}\ {}_r\Psi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z). $$

By utilizing the differential operator $\mathcal{Q}_{r,s}[a_1;q]$ defined in equation (1.19), we establish a new sufficient condition for the q-integral operator of meromorphic functions in Ω to belong to the class $\Omega_{N,q}(\eta)$.

Definition 1.1 Let $\mathcal{Q}_{r,s}[a_1;q]f(z)\in\Omega, c>0, |q|<1, \gamma_j>0$ for $j=1,2,\dots,n$, where $n\in\mathbb{N}$. We define the q- integral operator $\mathcal{H}_q(z):\Omega^n\to\Omega$ as follows:

$$ \mathcal{H}_q(z)=\frac{c}{z^{c+1}}\int_0^z u^{c-1}(u\mathcal{Q}_{r,s}[a_1;q]f_1(u))^{\gamma_1}\dots(u\mathcal{Q}_{r,s}[a_1;q]f_n(u))^{\gamma_n}d_q u. \tag{1.20} $$

Remark 1.1 if $r=1,s=0,a_1=q$ with $q\to 1$, the q-integral operator $\mathcal{H}_q(z)$ becomes the integral operator.

$$ H(z)=\frac{c}{z^{c+1}}\int_0^z u^{c-1}(uf_1(u))^{\gamma_1}\dots(uf_n(u))^{\gamma_n}du, $$

Introduced by Frasin [16]. If $c=1,r=1,s=0,a_1=q$ with $q\to 1$, the q-integral operator $\mathcal{H}_q(z)$ becomes the integral operator.

$$ H(z)=\frac{1}{z^2}\int_0^z (uf_1(u))^{\gamma_1}\dots(uf_n(u))^{\gamma_n}du,\quad (z\in\mathbb{U}^*), $$

Introduced by Mohammed and Darus [17]. If $n=1,\gamma_1=\gamma,f_1=f$, the integral operator $\mathcal{H}_q(z)$ becomes the integral operator.

$$ I_{q,c}^{\gamma}(z)=\frac{c}{z^{c+1}}\int_0^z u^{c-1}(u\mathcal{Q}_{r,s}[a_1;q]f(u))^{\gamma}d_q u,\quad (|q|<1,c,\gamma>0). $$

For $r=1,s=0,a_1=q,q\to 1,n=1,f_1=f,\gamma=1,$ then the integral operator is

$I_c(z)=\frac{c}{z^{c+1}}\int_0^z u^{c-1}f(u)du$, and it has been studied by various authors (cf.,eg., [18], [19], [20]).

Using the q-derivative, Lemma 2.4 from [21] can be extended to broader cases.

Lemma 1.1 *If $f(z) \in \Omega$ satisfies $f(z)D_q f(z) \neq 0$ in $\mathbb{U}$ and*

$$\Re e\left\{\frac{zD_q f(z)}{f(z)} - \frac{zD_q^2 f(z)}{D_q f(z)}\right\} < 2-\beta, \quad (z \in \mathbb{U}^*),$$

then:

$$-\Re e\left\{\frac{zD_q f(z)}{f(z)}\right\} < \frac{1}{3-2\beta}, \quad (z \in \mathbb{U}^*),$$

where $1/2 \leq \beta < 1$.

Proof. The function p is defined in $\mathbb{U}^*$as follows

$$-\frac{zD_q f(z)}{f(z)} = \delta + (1-\delta)p(z)$$

with $\delta = 1/(3-2\beta)$, applying a proof analogous to that of Theorem 2.2 in [21], we obtain

$$-\Re e\left\{\frac{zD_q f(z)}{f(z)}\right\} < \frac{1}{(3-2\beta)}, \quad (|q| < 1, z \in \mathbb{U}^*).$$

By sitting $q \to 1$ in Lemma1.1, we get Lemma 2.4 in [21].

Main results

Theorem 2.1 *Let $\mathcal{Q}_{r,s}[a_1;q]f_j(z) \in \Omega$ and $\gamma_j > 0$ for $j = 1,2,\ldots,n$, with*

$$1 < \sum_{j=1}^{n} [\gamma_j]_q < q^{-c}[c+1]_q, \quad (|q| < 1, c > 0). \tag{2.1}$$

If $\mathcal{H}_q(z) \in \Omega_q^*(\alpha)$ and $zD_q\mathcal{H}_q(z)/\mathcal{H}_q(z) \neq 0$ in $\mathbb{U}$, then $\mathcal{H}_q(z) \in \Omega_{N,q}(\eta)$, where $\eta > \frac{1}{q}$.

Proof. It follows from (1.20) that

$$D_q\left(\frac{z^{c-1}\mathcal{H}_q(z)}{c}\right) = z^{c+1}(z\mathcal{Q}_{r,s}[a_1;q]f_1(z))^{\gamma_1}\ldots(z\mathcal{Q}_{r,s}[a_1;q]f_n(z))^{\gamma_n} \tag{2.2}$$

Applying logarithmic q-differentiation as defined in (1.11) to (2.2) and then multiplying by z, we obtain.

$$\frac{z^2q^{c+2}D_q^2\mathcal{H}_q(z) + (q^{c+1}+cq^c+[c+1]_q)zD_q\mathcal{H}_q(z) + \frac{c}{q}[c+1]_q\mathcal{H}_q(z)}{zq^{c+1}D_q\mathcal{H}_q(z)+[c+1]_q\mathcal{H}_q(z)}$$

$$= \frac{c-1}{q} + \sum_{j=1}^{n} [\gamma_j]_q\left(\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)} + \frac{1}{q}\right),$$

which is equivalent to

$$\frac{\frac{zqD_q^2\mathcal{H}_q(z)}{D_q\mathcal{H}_q(z)} + \frac{c[c+1]_q}{q^{c+2}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)} + \left(\frac{1}{q^{c+1}}[c+1]_q + \frac{c}{q} + 1\right)}{1+\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}}$$

$$= \frac{c-1}{q} + \sum_{j=1}^{n} [\gamma_j]_q\left(\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)} + \frac{1}{q}\right).$$

Therefore, we have

$$\frac{-\left(\frac{zqD_q^2\mathcal{H}_q(z)}{D_q\mathcal{H}_q(z)}+1\right) - \frac{c[c+1]_q}{q^{c+2}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)} - \left(\frac{1}{q^{c+1}}[c+1]_q + \frac{c}{q}\right)}{1+\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}}$$

$$= \frac{1-c}{q} - \sum_{j=1}^{n} [\gamma_j]_q\left(\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)} + \frac{1}{q}\right). \tag{2.3}$$

From (2.3), we easily get

$$-\left(\frac{zqD_q^2\mathcal{H}_q(z)}{D_q\mathcal{H}_q(z)}+1\right) = \sum_{j=1}^{n} [\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)\left(1+\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)$$

$$-\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q} - \frac{1}{q}\right) + \frac{1}{q} + \frac{1}{q^{c+1}}[c+1]_q - \sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}. \tag{2.4}$$

Taking real part of both sides of equation (2.4), we get

$$-\Re e\left(\frac{zqD_q^2\mathcal{H}_q(z)}{D_q\mathcal{H}_q(z)}+1\right) =$$

$$\Re e\left[\sum_{j=1}^{n} [\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)\left(1+\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)\right]$$

$$+\frac{[c+1]_q}{q^{c+1}}\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}-\frac{1}{q}\right)\Re e\left(\frac{-\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}.$$

Thus, we have:

$$-\Re e\left(\frac{zqD_q^2\mathcal{H}_q(z)}{D_q\mathcal{H}_q(z)}+1\right)=$$

$$\Re e\left[\sum_{j=1}^{n}[\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)\left(1+\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)\right]$$

$$+\frac{[c+1]_q}{q^{c+1}}\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}-\frac{1}{q}\right)\Re e\left(-\frac{1}{\frac{zD_q\mathcal{H}_q(z)}{\mathcal{H}_q(z)}}\right)+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}$$

$$\leq\left|\sum_{j=1}^{n}[\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)\left(1+\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)\right|$$

$$+\frac{[c+1]_q}{q^{c+1}}\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}-\frac{1}{q}\right)\frac{\Re e\left(\frac{-zD_q\mathcal{H}_q(z)}{\mathcal{H}_q(z)}\right)}{\left|\frac{zD_q\mathcal{H}_q(z)}{\mathcal{H}_q(z)}\right|^2}+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}.$$

Let

$$\eta=\left|\sum_{j=1}^{n}[\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)\left(1+\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)\right|$$

$$+\frac{[c+1]_q}{q^{c+1}}\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}-\frac{1}{q}\right)\frac{\Re e\left(\frac{-zD_q\mathcal{H}_q(z)}{\mathcal{H}_q(z)}\right)}{\left|\frac{zD_q\mathcal{H}_q(z)}{\mathcal{H}_q(z)}\right|^2}+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}.$$

Since $\left|\sum_{j=1}^{n}[\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)\left(1+\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)\right|>0$, and $\mathcal{H}_q(z)\in\Omega_q^*(\alpha)$,

then we have

$$\eta>\frac{[c+1]_q}{q^{c+1}}\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}-\frac{1}{q}\right)\frac{\alpha}{\left|\frac{zD_q\mathcal{H}_q(z)}{\mathcal{H}_q(z)}\right|^2}+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}$$

$$>\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q},$$

Based on the hypothesis (2.1), it follows that $\eta>\frac{1}{q}$. Hence, $\mathcal{H}_q(z)\in\Omega_{N,q}(\eta)$, where $\eta>\frac{1}{q}$.

This concludes the proof.

Setting $r=1, s=0,$ and $\alpha_1=q$ with $q\to 1$ in Theorem 2.1 yields Theorem 2.1 as presented in [16]. Furthermore, by substituting $n=1, \gamma_1=\gamma,$ and $f_1=f$ in Theorem 2.1 leads to the following Corollary.

Corollary 2.2 *Let* $\mathcal{Q}_{r,s}[a_1;q]f(z)\in\Omega$ *and* $1<[\gamma]_q<q^{-c}[c+1]_q, c>0$. *If* $I_{q,c}^{\gamma}(z)\in\Omega_q^*(\alpha)$ *and* $zD_qI_{q,c}^{\gamma}(z)/I_{q,c}^{\gamma}(z)\neq 0$ *in* $\mathbb{U}$, *then* $I_{q,c}^{\gamma}(z)\in\Omega_{N,q}(\eta)$, *where* $\eta>\frac{1}{q}$.

Applying Lemma 1.1, we derive the following theorem.

Theorem 2.3 *Let* $\mathcal{Q}_{r,s}[a_1;q]f_j(z)\in\Omega$ *satisfies* $(\mathcal{Q}_{r,s}[a_1;q]f_j(z))(D_q\mathcal{Q}_{r,s}[a_1;q]f_j(z))\neq 0$ *in* $\mathbb{U}$ *for all* $j=1,2,\ldots,n, 1/2\leq\beta<1$, *and*

$$\Re e\left\{\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}-\frac{zD_q^2\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{D_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right\}<2-\beta,\quad (z\in\mathbb{U}^*).$$

If $\gamma_j>0$ for all $j=1,2,\ldots,n$, with

$$\sum_{j=1}^{n}[\gamma_j]_q<\frac{(3-2\beta)[c+1]_q}{q^{c-1}(3-q-2\beta)},\quad (|q|<1, c>0), \tag{2.5}$$

then $\mathcal{H}_q(z)\in\Omega_{N,q}(\eta)$, where $\eta>\frac{1}{q}$.

Proof. From equation (2.4) , we deduce:

$$-\left(\frac{zqD_q^2\mathcal{H}_q(z)}{D_q\mathcal{H}_q(z)}+1\right)=\left(\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)\left[\sum_{j=1}^{n}[\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)-\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}-\frac{1}{q}\right)\right]+$$

$$\sum_{j=1}^{n}[\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}. \tag{2.6}$$

Taking real part of both sides of equation (2.6), we get

$$-\Re e\left(\frac{zqD_q^2\mathcal{H}_q(z)}{D_q\mathcal{H}_q(z)}+1\right)=$$

$$\Re e\left\{\left(\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)\left[\sum_{j=1}^{n}[\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)-\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}-\frac{1}{q}\right)\right]\right\}$$

$$+\sum_{j=1}^{n}[\gamma_j]_q\Re e\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}$$

$$\le\left|\left(\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)\left[\sum_{j=1}^{n}[\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)-\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}-\frac{1}{q}\right)\right]\right|$$

$$+\sum_{j=1}^{n}[\gamma_j]_q\Re e\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}.$$

Let

$$\eta=\left|\left(\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)\left[\sum_{j=1}^{n}[\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)-\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}-\frac{1}{q}\right)\right]\right|$$

$$+\sum_{j=1}^{n}[\gamma_j]_q\Re e\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}. \qquad (2.7)$$

Then, applying Lemma1.1, and since

$$\left|\left(\frac{[c+1]_q}{q^{c+1}}\frac{\mathcal{H}_q(z)}{zD_q\mathcal{H}_q(z)}\right)\left[\sum_{j=1}^{n}[\gamma_j]_q\left(-\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f_j(z)}{\mathcal{Q}_{r,s}[a_1;q]f_j(z)}\right)-\left(\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}-\frac{1}{q}\right)\right]\right|>0,$$

we have:

$$\eta>\sum_{j=1}^{n}[\gamma_j]_q\left(\frac{1}{3-2\beta}\right)+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q-\sum_{j=1}^{n}\frac{[\gamma_j]_q}{q}=\sum_{j=1}^{n}[\gamma_j]_q\left(\frac{2\beta-3+q}{q(3-2\beta)}\right)$$

$$+\frac{1}{q}+\frac{1}{q^{c+1}}[c+1]_q.$$

Then, by the hypothesis (2.5), we have $\eta>\frac{1}{q}$. As result $\mathcal{H}_q(z)\in\Omega_{N,q}(\eta)$, where $\eta>\frac{1}{q}$.

Setting $r=1, s=0,$ and $\alpha_1=q$ with $q\to 1$ in Theorem 2.3 yields Theorem 2.3 as presented in [16]. Furthermore, by substituting$n=1,\gamma_1=\gamma, f_1=f$in Theorem 2.3 leads to the following Corollary.

Corollary 2.4 *Let $\mathcal{Q}_{r,s}[a_1;q]f(z)\in\Omega$ satisfies $(\mathcal{Q}_{r,s}[a_1;q]f(z))(D_q\mathcal{Q}_{r,s}[a_1;q]f(z))\neq 0$ in $\mathbb{U}$,$1/2\le\beta<1$, and*

$$\Re e\left\{\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f(z)}{\mathcal{Q}_{r,s}[a_1;q]f(z)}-\frac{zD_q^2\mathcal{Q}_{r,s}[a_1;q]f(z)}{D_q\mathcal{Q}_{r,s}[a_1;q]f(z)}\right\}<2-\beta \quad (z\in\mathbb{U}^*).$$

If $[\gamma]_q<\frac{(3-2\beta)[c+1]_q}{q^{c-1}(3-q-2\beta)}, |q|<1, c>0$, then $I_{q,c}^{\gamma}(z)\in\Omega_{N,q}(\eta)$, where $\eta>\frac{1}{q}$.

By putting $\beta=1/2$ in Corollary 2.4 gives the following result.

Corollary 2.5 *Let $\mathcal{Q}_{r,s}[a_1;q]f(z)\in\Omega$ satisfies $(\mathcal{Q}_{r,s}[a_1;q]f(z))(D_q\mathcal{Q}_{r,s}[a_1;q]f(z))\neq 0$ in $\mathbb{U}$,and*

$$\Re e\left\{\frac{zD_q\mathcal{Q}_{r,s}[a_1;q]f(z)}{\mathcal{Q}_{r,s}[a_1;q]f(z)}-\frac{zD_q^2\mathcal{Q}_{r,s}[a_1;q]f(z)}{D_q\mathcal{Q}_{r,s}[a_1;q]f(z)}\right\}<3/2 \quad (z\in\mathbb{U}^*).$$

If $[\gamma]_q<\frac{2[c+1]_q}{q^{c-1}(2-q)}$, $|q|<1, c>0$, then $I_{q,c}^{\gamma}(z)\in\Omega_{N,q}(\eta)$, where $\eta>\frac{1}{q}$.

Utilizing equation (1.15) and equation (2.7), the following theorem can be established

Theorem 2.6 *Let $\mathcal{Q}_{r,s}[a_1;q]f_j(z)\in\Omega$ and $\gamma_j>0$ for $j=1,2,\ldots,n$, with*

$$\sum_{j=1}^{n}[\gamma_j]_q<\frac{[c+1]_q}{q^c(1-q\alpha)}, \quad (|q|<1, c>0, 0\le\delta<1). \qquad (2.8)$$

If $\mathcal{Q}_{r,s}[a_1;q]f_j(z)\in\Omega_q^*(\alpha)$, then $\mathcal{H}_q(z)\in\Omega_{N,q}(\eta)$, where $\eta>\frac{1}{q}$.

Substituting $n=1,\gamma_1=\gamma,$ and $f_1=f$ in Theorem 2.6 leads to the following Corollary.

Corollary 2.7 *Let $\mathcal{Q}_{r,s}[a_1;q]f(z)\in\Omega$ and $[\gamma]_q<\frac{[c+1]_q}{q^c(1-q\alpha)}, |q|<1, c>0,$ where $0\le\delta<1$. If $\mathcal{Q}_{r,s}[a_1;q]f(z)\in\Omega_q^*(\alpha)$, then $I_{q,c}^{\gamma}(z)\in\Omega_{N,q}(\eta)$, where $\eta>\frac{1}{q}$.*

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