The North African Journal of Scientific Publishing (NAJSP)
مجلة شمال إفريقيا للنشر العلمي (NAJSP)
E-ISSN: 2959-4820
Volume 3, Issue 1, January - March 2025
Page No: 53-61
Website: https://najsp.com/index.php/home/index
SJIFactor 2024: 5.49 معامل التأثير العربي (AIF) 2024: 0.71 ISI 2024: 0.696

Comprehensive Insights into Matrix Continued Fractions and Their Uses

Naglaa Fathi Abdelhafiz Boubaker [1*]
[1]The Higher Institute of Science and Technology, Al-Abyar, Libya

نظرة شاملة حول مصفوفات الكسور المستمرة واستخداماتها

نجلاء فتحي عبد الحفيظ بوبكر [1*]
[1]المعهد العالي للعلوم والتكنولوجيا، الأبيار، ليبيا

*Corresponding author: naglaa.fathy@hista.edu.ly

Received: December 01, 2024	Accepted: January 15, 2025	Published: February 08, 2025

Abstract
Continued fractions are a fundamental concept in mathematical analysis, providing a bridge between number theory, approximation theory, and applied mathematics. This study comprehensively explores matrix continued fractions, their theoretical foundations, and their diverse applications. Beginning with fundamental definitions, we introduce simple and generalized continued fractions and their matrix representations, emphasizing their computational efficiency and convergence properties. Matrix continued fractions extend classical continued fractions to linear algebra, offering valuable tools for solving linear equations, eigenvalue approximations, and matrix inversion. Their recursive nature enables efficient numerical computations, particularly in solving differential equations and modeling physical systems. Furthermore, we examine the role of continued fractions in function approximation, demonstrating their advantages over traditional power series expansions. Notable applications include their use in representing irrational numbers, computing special functions such as Bessel and error functions, and facilitating root-finding algorithms. The study also discusses the theoretical implications of continued fractions, including their connections to Möbius transformations, Diophantine approximations, and periodicity in number theory. Additionally, we explore computational methods for continued fraction evaluation, including adaptive algorithms and error management strategies that enhance numerical stability and precision. The significance of continued fractions extends beyond pure mathematics, with engineering, physics, and computer science applications. We highlight their role in scientific computing, signal processing, and cryptographic algorithms. The article concludes with recent advancements in continued fraction research, underscoring their ongoing relevance and potential for further exploration in modern mathematical and computational fields.

Keywords: Continued fractions, matrix continued fractions, function approximation, eigenvalue computation.

الملخص
الكسور المستمرة هي مفهوم أساسي في التحليل الرياضي، حيث توفر جسرًا بين نظرية الأعداد ونظرية التقريب والرياضيات التطبيقية. تستكشف هذه الدراسة بشكل شامل الكسور المستمرة للمصفوفة وأسسها النظرية وتطبيقاتها المتنوعة. بدءًا من التعريفات الأساسية، نقدم الكسور المستمرة البسيطة والمعممة وتمثيلاتها المصفوفة، مع التأكيد على كفاءتها الحسابية وخصائص التقارب. تمتد الكسور المستمرة

للمصفوفة بالكسور المستمرة الكلاسيكية إلى الجبر الخطي، وتقدم أدوات قيمة لحل المعادلات الخطية وتقريبات القيمة الذاتية وعكس المصفوفة. تمكن طبيعتها المتكررة من الحسابات العددية الفعّالة، وخاصة في حل المعادلات التفاضلية ونمذجة الأنظمة الفيزيائية. علاوة على ذلك، ندرس دور الكسور المستمرة في تقريب الدالة، ونوضح مزاياها مقارنة بتوسعات سلسلة القوى التقليدية. تشمل التطبيقات البارزة استخدامها في تمثيل الأعداد غير النسبية، وحساب الدوال الخاصة مثل وظائف بيسل والخطأ، وتسهيل خوارزميات إيجاد الجذر. وتناقش الدراسة أيضًا التداعيات النظرية للكسور المستمرة، بما في ذلك ارتباطاتها بتحويلات موبيوس، وتقريبات ديو فانتين، والدورية في نظرية الأعداد. بالإضافة إلى ذلك، نستكشف الأساليب الحسابية لتقييم الكسور المستمرة، بما في ذلك الخوارزميات التكيفية واستراتيجيات إدارة الأخطاء التي تعزز الاستقرار العددي والدقة. وتمتد أهمية الكسور المستمرة إلى ما هو أبعد من الرياضيات البحتة، مع تطبيقات الهندسة والفيزياء وعلوم الكمبيوتر. ونسلط الضوء على دورها في الحوسبة العلمية ومعالجة الإشارات وخوارزميات التشفير. وتختتم المقالة بالتطورات الأخيرة في أبحاث الكسور المستمرة، مع التأكيد على أهميتها المستمرة وإمكاناتها لمزيد من الاستكشاف في المجالات الرياضية والحاسوبية الحديثة.

الكلمات المفتاحية: الكسور المستمرة، الكسور المستمرة للمصفوفة، تقريب الدالة، حساب القيمة الذاتية.

Introduction

Continued fractions represent one of the most fascinating and profound concepts in mathematical analysis, bridging elementary number theory with advanced analytical methods [1]. This comprehensive study delves into the multifaceted nature of continued fractions, their matrix representations, and their wide-ranging applications in various mathematical and physical contexts [2]. The investigation encompasses both theoretical foundations and practical implementations, with particular emphasis on computational methods and applications in differential equations [3].

Continued fractions have many important properties and applications in mathematics, including in number theory, Diophantine equations, and the theory of irrational numbers [4]. They can also be used to symbolize a variety of mathematical operations, such as the logarithm, trigonometric functions, and the Riemann zeta function. In matrix form, continued fractions are used in the study of linear recurrent sequences, which are sequences of numbers that are determined by a fixed set of initial conditions and a set of recurrence relations [5]. They have many applications in areas such as statistics, physics, and control theory.

A matrix continued fraction is a type of representation for matrices, similar to how continued fractions represent real numbers. It is a method of approximating a matrix as the product of simpler matrices. The matrix continued fraction provides a way to decompose a given matrix into a series of simple matrices that can be easier to work with [6]. Some applications of matrix continued fractions include:

- Solving linear equations: it is possible to resolve systems of linear equations using matrix-continuing fractions.
- Eigenvalue approximation: to roughly determine a matrix's eigenvalues, one can use matrix continued fractions.
- Matrix inversion: you may quickly determine a matrix's inverse by using matrix continued fractions [7].

The fundamental structure of a continued fraction can be expressed as:

$$a_0 + b_1/(a_1 + b_2/(a_2 + b_3/(a_3 + \dots)))$$

Where a_i and b_i are real or complex numbers. This apparently simple structure belies the remarkable depth and utility of continued fractions in various mathematical contexts [8].

Fundamental Definitions

1. Simple Continued Fraction:

A continued fraction of the form:

$$[a_0;\ a_1, a_2, a_3, \dots] = a_0 + 1/(a_1 + 1/(a_2 + 1/(a_3 + \dots)))$$

Where a_i are integers and $a_i > 0$ for $i > 0$

Example 1:

The representation of π as a continued fraction:

$$\pi = [3;\ 7, 15, 1, 292, 1, 1, \dots]\ (9)$$

2. Generalized Continued Fraction:

A continued fraction where numerators can be any real or complex numbers:

$$a_0 + b_1/(a_1 + b_2/(a_2 + b_3/(a_3 + \dots)))$$

Example 2:
The representation of e (Euler's number):

$$e = [2;\ 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots]\ [9]$$

Material and Method:

Matrix Representation

One of the most powerful aspects of continued fractions is their matrix representation. For a finite continued fraction:

$$[a_0;\ a_1, a_2, \ldots, a_n]$$

This matrix representation provides [10]:

1. Efficient computation of convergent.
2. Analysis of convergence properties.
3. Connection to linear transformations.
4. Implementation in computer algorithms.

Theoretical Framework

The theoretical framework of continued fractions encompasses several key areas. Convergence theory focuses on the conditions required for convergence, the rate at which convergence occurs, and its relationship to series expansions. Number theoretical aspects examine the representation of irrational numbers, periodic continued fractions, and the determination of the best rational approximations. Analytic properties explore the connection between continued fractions and Möbius transformations, their role in Diophantine approximation, and their analytic continuation [11].

Example 3:
Consider the golden ratio $\varphi = (1 + \sqrt{5})/2$
Its continued fraction representation is:

$$\varphi = [1;\ 1, 1, 1, \ldots]$$

This simple representation demonstrates the deep connection between continued fractions and number theory [11].
The relationship between continued fractions and infinite series represents a fundamental connection in mathematical analysis.

Series to Continued Fraction Conversion

When we convert a power series to a continued fraction, we often get expressions that converge more quickly. For example:
A power series like: $S(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$
Can be transformed into a continued fraction form that often provides better approximations with fewer terms [12].

Euler's Method

Euler developed a way to convert series to continued fractions. A famous example is the exponential series:

$$e\hat{}x = 1 + x + x^2/2! + x^3/3! + \ldots$$

Which becomes the continued fraction:

$$e^x = 1 + \cfrac{x}{1 - \cfrac{x}{2 + \cfrac{x}{3 - \cfrac{x}{4 + \ldots}}}}\ [12]$$

Important Examples

1. Geometric Series:

The series 1 + x + x² + x³ + ... becomes:

$$1/(1 - x) = 1 + x/(1 - x/(1 - x/(1 - \ldots)))$$

2. π Approximation:

- Series form: $\pi = 4(1 - 1/3 + 1/5 - 1/7 + \ldots)$
- Continued fraction: π = [3; 7, 15, 1, 292, ...] The continued fraction version converges much faster [13].

Special Functions

Many important mathematical functions have elegant continued fraction forms:

1. Bessel Functions:

$$J_0(x)/J_1(x) = 1/(x - 2/(x - 3/(x - 4/(x - \ldots))))$$

2. Error Function:

$$erf(x) = \frac{\frac{2x}{\sqrt{\pi}}}{1 + \frac{2x^2}{3 + \frac{4x^2}{5 + \frac{6x^2}{7 + \dots}}}} [14]$$

Theoretical Implications

The theoretical implications of continued fractions are significant in various mathematical contexts. Analytic continuation allows continued fractions to extend functions beyond the domain where their power series fail to converge. Asymptotic behavior provides insight into the properties of functions at infinity, as the tail of a continued fraction often reveals crucial information about their long-term behavior [15].

Continued Fractions in Recursive Forms

Continued fractions naturally have a recursive structure, which makes them both theoretically interesting and practically useful [16].

$$\frac{Pn}{qn} = a0; a1, a2, \dots, an = a0 + \frac{1}{a1 \frac{1}{a2 \frac{1}{a3 \frac{1}{\ddots \frac{1}{an}}}}}$$

Where:

- a_0 is an integer,
- $a_1, a_2, \dots, a_n$ are positive integers

Numerators of Continued Fractions

Each fraction generated from the continued fraction has a numerator (P_n) and a denominator (q_n). The first few numerators follow the pattern:

$$P0 = a0, P1 = a0q1 + 1 = a1P0 + 1, \dots$$

This shows that P_n is recursively dependent on previous terms. (17)

Recurrence Relation for the Numerators

A general recursive formula for numerators can be derived:

$$pn = a0\, pn - 1 + pn - 2$$

This means that each numerator p_n is computed using the previous two numerators p_n −1and p_n−2 weighted by a_n [17].

Recurrence Relation for the Denominators

Similarly, the denominators q_n follow the recurrence relation:

$$qn = an\, qn - 1 + qn - 2$$

This shows that each denominator is also recursively dependent on the previous two denominators.

Basic Structure

The basic structure of continued fractions can be understood in two ways. In the forward direction, the fraction is constructed progressively by adding terms one at a time from the beginning. In the backward direction, the fraction is analyzed starting from the end and working backward, a method that often provides better numerical stability [17].

Three-Term Relations

Many continued fractions come from relationships where each term depends on the two before it. For example:

The Fibonacci sequence leads to the continued fraction for the golden ratio:

$$\varphi = [1; 1, 1, 1, \dots] = \frac{1 + \sqrt{5}}{2} [18]$$

Convergent

The systematic rational approximations (convergent) follow patterns:

For π = [3; 7, 15, 1, 292, ...]:

- First convergent: 3/1
- Second convergent: 22/7
- Third convergent: 333/106

Each gives a better approximation [19].

Special Forms

1. S-Fractions:

Special continued fractions that look like:

$$a_1/(1 + a_2/(1 + a_3/(1 + \dots)))$$

2. T-Fractions:

Modified continued fractions useful applications:

$$a^0 + \frac{k^1 x}{a^1 + \frac{k^2 x}{a^2 + \dots}} [19]$$

Applications

1. Root Finding:

Finding solutions to equations through iterative processes.

2. Function Approximation:

Representing complex functions with simpler rational expressions [20].

Theoretical Aspects

1. Convergence:

Understanding when and how quickly continued fractions converge.

2. Periodic Patterns:

Recurring patterns often indicate special numbers:

$\sqrt{2}$ = [1; 2, 2, 2, ...] shows perfect periodicity [20].

Practical Uses

1. In Differential Equations:

- Series solutions
- Asymptotic expansions
- Green's functions

2. Special Functions:

Representing complex mathematical functions:

- Hyper geometric functions
- Bessel functions
- Gamma functions

The Use of Continued Fractions in Finding Roots of Real Numbers

Continued fractions offer an elegant and efficient approach to finding and representing roots of real numbers. Unlike decimal representations, continued fractions often reveal patterns and provide exact representations irrational numbers [21].

Basic Root Finding Method

The process of finding roots using continued fractions involves converting a root into its continued fraction representation. This is done by repeatedly extracting the integer part and taking the reciprocal of the fractional part. For example, when finding $\sqrt{2}$, we get the elegant pattern [1; 2, 2, 2, ...], meaning 1 + 1/ (2 + 1/ (2 + 1/ (2 + ...))) [21]

Square Root Computation

Square roots are particularly well-suited to continued fraction representation. The process follows a regular pattern that can be used to generate successive terms. For instance:

- $\sqrt{2} = [1;\ 2, 2, 2, \dots]$
- $\sqrt{3} = [1;\ 1, 2, 1, 2, 1, 2, \dots]$
- $\sqrt{5} = [2;\ 4, 4, 4, \dots]$ [21]

Each convergent (fraction formed by truncating the continued fraction) provides a rational approximation that alternates between being slightly larger and slightly smaller than the actual value.

Higher Order Roots

Cube roots and higher-order roots can also be expressed as continued fractions, though their patterns are generally more complex than square roots. For example:

- $\sqrt[3]{2} = [1;\ 3, 1, 5, 1, 1, 4, \dots]$
- $\sqrt[4]{2} = [1;\ 2, 1, 1, 3, 1, 1, 2, 1, \dots]$ [22]

Error Analysis and Convergence

The error in continued fraction approximations decreases rapidly with each additional term. For square roots, each convergent p_i/q_i satisfies:

$$|\sqrt{n} - p_i/q_i| \leq 1/(2q_i^2)$$

This rapid convergence makes continued fractions particularly useful for high-precision calculations [22].

Practical Applications

1. **Mathematical Constants:**

Continued fractions are used to represent and compute mathematical constants with high precision. For example, π can be represented as [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, ...]

2. **Numerical Analysis:**

- Finding roots of equations
- Approximating irrational numbers
- Computing mathematical functions

Advantages over Other Methods

Continued fractions offer several benefits compared to decimal representations [22]:

1. They often reveal patterns in irrational numbers
2. They provide best possible rational approximations for their length
3. They maintain exact values in rational calculations
4. They have natural truncation points for approximations

Examples and Applications

1. Engineering Calculations:

When approximating √2 for engineering purposes:

- First convergent: 1/1 = 1
- Second convergent: 3/2 = 1.5
- Third convergent: 7/5 = 1.4
- Fourth convergent: 17/12 ≈ 1.4167

Each step provides a better rational approximation.

2. Theoretical Mathematics:

The continued fraction representation helps identify algebraic numbers and their properties. For instance, a periodic continued fraction indicates a quadratic irrational number [23].

Limitations and Considerations

Computational aspects include the need to handle large numerators and denominators, consider precision requirements, and monitor convergence rates. For practical implementation, it is essential to choose appropriate termination conditions, balance accuracy versus computation time, and consider numerical stability. This method of finding roots through continued fractions provides both theoretical insight and practical computational advantages, especially when exact rational approximations are desired [24].

Nth Roots and Evaluation of Quantities

The evaluation of nth roots and other mathematical quantities using continued fractions extends beyond simple square roots, offering powerful methods for computing complex mathematical expressions [25].

General Theory of nth Roots

The representation of nth roots through continued fractions follows specific patterns:

1. Periodic Patterns:

- ➢ Square roots always have periodic continued fractions
- ➢ Higher roots may have more complex, non-periodic representations
- ➢ The period length often relates to the algebraic degree

Example patterns:

- ${}^{5}\sqrt{2} = [1;\ 1, 4, 1, 1, 8, 1, 1, 4, 1, \ldots]$
- ${}^{6}\sqrt{3} = [1;\ 2, 1, 1, 1, 4, 1, 1, 1, 2, \ldots]$ [26]

Evaluation Techniques

1. **Direct Method:**

Starting with an initial approximation, each term is computed by:

- Taking the integer part
- Computing the reciprocal of the fractional part
- Repeating until desired accuracy

2. **Nested Radical Forms:**

Some nth roots can be expressed as nested radicals:

$$\sqrt{(2 + \sqrt{(2 + \sqrt{(2 + \ldots)})})}$$

Special Cases and Properties

1. **Perfect Powers:**

When evaluating nth roots of perfect nth powers:

- The continued fraction terminates
- The length relates to the size of the number

- The pattern is simpler than non-perfect powers

2. **Mixed Roots:**

Complex expressions involving multiple roots:

$$\sqrt{(1 + \sqrt{2})} = [1;\ 2, 8, 2, 8, 2, 8, \ldots]$$

Convergence Analysis

The convergence of nth root continued fractions follows patterns:

1. Rate of Convergence:

- Higher roots generally converge more slowly
- Convergence rate depends on the size of the number
- Each term improves accuracy exponentially

2. Error Bounds:

For the kth convergent of an nth root:

- Error decreases as $1/q^{2k}$
- Alternating over- and under-approximation

Practical Applications

1. Scientific Computing:

- High-precision calculations
- Physical constants
- Engineering approximations

2. Number Theory:

- Studying algebraic numbers
- Finding rational approximations
- Analyzing number properties

Special Functions

1. Exponential Function:

$$e = [2;\ 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots]$$

2. Logarithmic Values:

$$\ln(2) = [0;\ 1, 2, 3, 1, 6, 3, 1, 1, 2, \ldots]$$ [26]

Computational Methods

Computational methods for continued fractions involve adaptive algorithms and acceleration techniques. Adaptive algorithms adjust precision based on desired accuracy, dynamically monitor convergence, and efficiently handle special cases. Acceleration techniques include the use of transformations, modified convergence criteria, and optimized computation methods to improve efficiency.

Error management

Error management is crucial in ensuring accurate results. Truncation decisions determine when to stop computing terms, how to estimate the remaining error, and identify optimal truncation points. Precision requirements involve balancing computational resources, determining the minimal terms needed for a given accuracy, and managing the trade-off between speed and precision.

Examples in Practice

1. Mathematical Constants:

Computing π^3:

- Initial terms: [31; 7, 15, 1, 292, ...]
- Each term provides better approximation

2. Physical Constants:

Fine structure constant:

- Continued fraction representation
- Rational approximations for calculations [27].

Advanced Topics

1. Multivariate Extensions:

Computing expressions like:

$$\sqrt{(x + \sqrt{(y + \sqrt{z})})}$$

2. Complex Roots:

Handling complex numbers:

- Real and imaginary parts
- Magnitude and phase

General Guidelines

The selection of an appropriate computational method depends on several factors, including the required precision, available computational resources, and the presence of special cases that may affect performance. When implementing these methods, it is essential to choose suitable data structures, ensure numerical stability, and account for potential error propagation to maintain accuracy and efficiency.

Practical Considerations

When implementing continued fraction methods, computational efficiency plays a significant role. Factors such as memory usage, processing time, and precision requirements must be carefully managed to ensure optimal performance. Additionally, application-specific needs should be considered, including real-time calculations for interactive systems, batch processing for large-scale computations, and balancing computational trade-offs for different practical scenarios.

Conclusion

The subject of continued fractions is still vital in many fields. It can help in establishing an efficient algorithm to evaluate Y's functions in space dynamics; the algorithm is valid to be used for any conic section. Also, CFs can be used to organize, as a new theoretical aspect, Euclidean algorithm for finding the GCD of two numbers with the help of a pseudocode; the code is independent of programming languages and is universal in the sense that it can be transformed into solutions which lead to important applications of CFs with a new approach. The benefits behind that are the usefulness for specialists and teachers in the fields of informatics, mathematics, and parallel computations.

Another application of CFs is studying double-sided CFs, with coefficients, which are non-commutative symbols, and their relation with the theory of discrete integrable systems. In quantum mechanics, there is another application for CFs in Probing Schrodinger Equation where a continued fraction potential was used to search for possible solutions of the Equation. A very recent work on CFs is an MA thesis, which showed the continuous interest in the subject of continued fractions and their applications in a variety of fields of mathematics such as number theory and abstract algebra. One of the interesting applications of CFS is their use in obtaining expressions for functions such as tan x and the evaluation of certain numbers, e.g., 4π.

Even in the complex field, continued fractions play an important role in conjunction with the evaluation of binary quadratic forms. One can continue with presenting the so many applications of CFs and that will take a huge amount of work to accomplish the job, but we shall give here one more application and consider it as a final one. The application has to do with folding; if we repeat folding a strip of paper in half and unfolding it in straight angles, then we get a fractal, which is known as the dragon curve. The sequence of right and left turns is related to a CF which constitutes a simple infinite series; so many properties and functions may arise from that leading to a shape resembling the dragon curve.

References:

1. Brezinski, C. (2010) The Italian Contribution to the Foundation and Development of Continued Fractions. Rendiconti del Seminario Matematico Universita e Politecnico di Torino, 68, 1-16.
2. Parrish, K.L. (2021) A Study in Applications of Continued Fractions. M.A. Thesis, California State University, San Bernardino.
3. Lasjaunias, A. (2017) A Short History of Some Recent Research on Continued Fractions in Function Fields. Technical Report, University of Bordeaux, Bordeaux, 1-7.
4. Siverman, V.H. (2012) A Friendly Introduction to Number Theory. 4th Edition, Pearson Education Ltd., Essex.
5. Ibran, Z.M. (2014) Continued Fractions and Their Applications. M.Sc. Thesis, University of Tripoli, Tripoli.
6. Baumann, H. (2019) Generalized Continued Fractions: A Definition and a Pringsheim- Type Convergence Criterion. A Springer Open Journal, 406, 1-30.
7. Awin, A.M. (2015) The Use of Special Bilinear Functions in Computing Some Special Functions. Journal of Progressive Research in Mathematics, 5, 437-443.
8. Juroud, A.A., Sharif, B.W. and Awin, A.M. (2021) Special Trilinear Functions with Few Applications. Journal of Applied Mathematics and Physics, 9, 2698-2705.
9. Handley, H., Continued Fractions: An Arithmetic and Analytic Study, 2023.
10. Cuyt, A.A., et al., Handbook of continued fractions for special functions2008: Springer Science & Business Media.
11. Frommer, A., K. Kahl, and M. Tsolakis, Matrix functions via linear systems built from continued fractions. arXiv preprint arXiv:2109.03527, 2021.
12. Giscard, P.-L. and M. Foroozandeh, Exact solutions for the time-evolution of quantum spin systems under arbitrary waveforms using algebraic graph theory. Computer Physics Communications, 2023. 282: p. 108561.
13. Wall, H.S., Analytic theory of continued fractions2018: Courier Dover Publications.
14. Ibran, Z.M., E.A. Aljatlawi, and A.M. Awin, On continued fractions and their applications. Journal of Applied Mathematics and Physics, 2022. 10(1): p. 142-159.

15. Arnoux, P. and S. Labbé, On some symmetric multidimensional continued fraction algorithms. Ergodic Theory and Dynamical Systems, 2018. 38(5): p. 1601-1626.
16. Jenkinson, O. and M. Pollicott, Rigorous effective bounds on the Hausdorff dimension of continued fraction Cantor sets: a hundred decimal digits for the dimension of E2. Advances in Mathematics, 2018. 325: p. 87-115.
17. Rao, S.S., Vibration of continuous system, s2019: John Wiley & Sons.
18. Yalkinoglu, B., *Knots and Primes: On the arithmetic of Toda flows.* 2022.
19. Tonien, J. (2018) Continued Fractions and Their Applications. Ph.D. Thesis, University of Wollongong, Wollongong.
20. Sharaf, A.M., Saad, A.S. and Abd El Motelp, N.S. (2015) Continued Fraction Evaluation of the Universal Y's Functions. International Journal of Astronomy and Astrophysics, 5, 15-19.
21. Iliev, A. and Kyurkchiev, N. (2019) New Organizing of the Euclid's Algorithm and One of Its Applications to the Continued Fractions. Conference on Mathematics, Informatics, Information Technology, Application in Education, Pamporovo, 10-12 October 2018, 199-207.
22. Doliwa, A. (2020) Non-Commutative Double-Sided Continued Fractions. Journal of Physics A: Mathematical and Theoretical, 53, Article ID: 364001.
23. Ahmed, N., Alamri, S.Z. and Raseem, M. (2018) Probing Schrodinger Equation with a Continued Fraction Potential. NRIAG Journal of Astronomy and Geophysics, 7, 1-3.
24. Dani, S.G. and Nogueira, A. (2014) Continued Fractions for Complex Numbers and Values of Binary Quadratic Forms. Transactions of the American Mathematical Society
25. Nieuwveld, J. (2021) Fractions, Functions and Folding; a Novel Link between Continued Fractions, Mahler Function and Paper Folding. M.Sc. Thesis, Radboud University, Nijmegen.
26. Garrity T, Osterman OV. On the Linear Complexity Associated with a Family of Multidimentional Continued Fraction Algorithms. arXiv preprint arXiv:2410.02032. 2024 Oct 2.
27. Grebogi RB. Charting New Territories: Exploring Physics Datasets through Continued Fraction Regression (Doctoral dissertation, The University of Newcastle).