The North African Journal of Scientific Publishing (NAJSP)
مجلة شمال إفريقيا للنشر العلمي (NAJSP)
E-ISSN: 2959-4820
Volume 2, Issue 3, July – September 2024, Page No: 118-123
Website: https://najsp.com/index.php/home/index
SJIFactor 2023: 3.733 معامل التأثير العربي (AIF) 2023: 0.63 ISI 2023: 0.383

A Remark on Geometrical Approach to Term Orders for *n* = 2

Samira Ziada [1*], Entisar El-Yagubi [2], Zaynab Ahmed Khalleefah [3]
[1,2,3] Department of Mathematics, Faculty of Science, Gharyan University, Gharyan, Libya

سميرة زيادة [1*]، انتصار اليعقوبي [2]، زينب أحمد خليفة [3]
[3،2،1] قسم الرياضيات، كلية العلوم، جامعة غريان، غريان، ليبيا

*Corresponding author: samira.ziada@gu.edu.ly

Received: June 16, 2024	Accepted: August 09, 2024	Published: August 17, 2024

Abstract:
The division algorithm for multivariate polynomials over fields has been introduced not very long ago, in connection with algorithmic and computational problems in the rings since the work of Buchberger. The well-known fact is a term order must be a well-ordering and the division procedure in the ring of multivariate polynomials over a field terminates even if the division term is not leading term, but is freely chosen. In this paper, we will show the original geometric approach to the classification of all possible orders on the ring of polynomials in two variables is given. The connection between this classification and the well-known classification of Robiano is exposed in details.

Keywords: Term ordering, Gröbner basis, Gröbner fan.

الملخص
تم تقديم خوارزمية القسمة لكثيرات الحدود في عدة متغيرات على الحقول منذ فترة ليست بالطويلة فيما يتعلق بالمشكلات الخوارزمية والحسابية على الحلقات حيث عمل عليها بوخبرجر. والحقيقة المعروفة جيدا هي ان ترتيب الحدود يجيب ان يكون ترتيبا جيدا وأن اجراء عملية القسمة في حلقات كثيرات الحدود في عدة متغيرات على حقل ما تنتهي حتى لو لم يكن حد القسمة هو الحد الاعلى ولكن تم اختياره بحرية. في هذه الورقة سوف نعرض ونبين النهج الهندسي الأصلي لجميع الاحتمالات الممكنة لتصنيف او ترتيب الحدود على حلقات كثيرات الحدود في متغيرين وعرض العلاقة بين هذا التصنيف والتصنيف المعروف لروبيانو بالتفصيل.

الكلمات المفتاحية: ترتيب الحدود، أساس قربنر، مروحة قربنر.

1. Term orderings

The most important ingredient in the algorithm of multivariate polynomial division is monomial or term order. Let us recall what a term order is. For more details the reader could refer to ([1], [3], [10]). Here, the set of nonnegative integers is denoted by $\mathbb{N}_0$.

Definition 1. *Let K be a field. A term ordering on* $K^0[x_1,\ldots,x_n]$ *is any partial order relation* $\prec$ *on* $\mathbb{N}_0^n$ *(or equivalently, any partial order relation on the set of monomials*

$x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$, $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathbb{N}_0^n$*) such that:*

1. $\prec$ *is a total (linear) ordering on* $\mathbb{N}_0^n$ *(this means that every two elements are comparable);*

2. If $\alpha \prec \beta \in \mathbb{N}_0^n$ *and* $\gamma \in \mathbb{N}_0^n$ *then* $\alpha+\gamma \prec \beta+\gamma$ *(the additive property);*

3. $\prec$ *is a will-ordering on* $\mathbb{N}_0^n$ *(this means that every nonempty subset of* $\mathbb{N}_0^n$ *has a smallest element under* $\prec$*).*

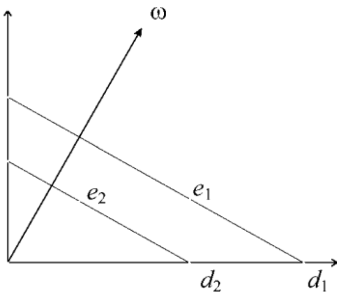


Figure 1

In this paper, the case $n = 2$ will be considered. In general, Sturmfels [2] gives a way to describe term orders by using weight vectors and arbitrary term orders as tiebreakers. In the case of two variables thereis a simpler method to express these weight vectors as line slopes. So, the weight vector $\omega = (p, q)$ is converted to $\frac{q}{p}$, where $p \neq 0$ $(and\ \infty\ if\ p = 0)$. The slope m represents the weight vector $\omega_m = (1, m)$ and the corresponding family of parallel lines $x + my = d$. We start with two important propositions about irrational slopes.

Proposition 1. *Any positive irrational number m determines a term order.*
Proof. Let $\omega = \omega_m = (1\ ,\ m)$ with m irrational, and choose an arbitrary term order $\prec$. Then we can compare any two exponent vectors $e_1 = (a_1, b_1)$, and $e_2 = (a_2, b_2)$ using $\prec_m$. If we define term order $\prec_m$ on $K[x_1, \ldots, x_n]$ for nonzero weight vector by $x^\alpha \prec_m y^\beta$ if $\alpha \cdot \omega \prec \beta \cdot \omega$ or if $\alpha \cdot \omega = \beta \cdot \omega$ and $x^\alpha \prec y^\alpha$, then $e_1 \prec_m e_2 \Leftrightarrow e_1 \cdot \omega_m \prec e_2 \cdot \omega_m$ or $e_1 \cdot \omega_m = e_2 \cdot \omega_m$ and $e_1 \prec e_2$. But m is irrational and $a_1 + b_1 m \neq a_2 + b_2 m$ (since $a_1 + b_1 m = a_2 + b_2 m$ would imply $m = \frac{a_1 + a_2}{b_2 - b_1} \in \mathbb{Q}$),
$e_1 \prec_m e_2 \Leftrightarrow a_1 + b_1 m \prec a_2 + b_2 m \Leftrightarrow e_1 . \omega_m \prec e_2 . \omega_m$.
The vector ω_m determines a family of lines (x, y). $\omega_m = d$ or $x + my = d$ with different d's.
Since m is irrational, every such line can contain at most one point from $\mathbb{Z}^2$. The relation $e_1 \prec_m e_2$ means that points e_1 and e_2 lay on different lines with respective parameters $d_1 \prec d_2$.
One can immediately see that the resulting order $\prec_m$ for irrational m does not depend on the choice of the original order $\prec$ in $\mathbb{Z}^2$

Proposition 2. Different numbers give different term orders.
Proof. Take two positive numbers $m_1 \neq m_2$, then there exists a rational $\frac{p}{q}$ such that $m1 \prec \frac{p}{q} \prec m_2$. Take $\omega_1 = (1, m_1)$, $\omega_2 = (1, m_2)$

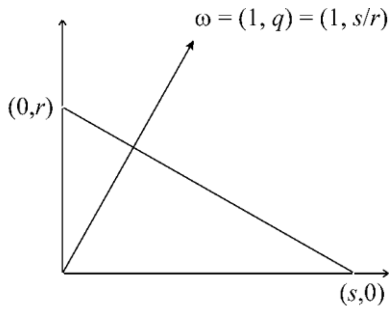
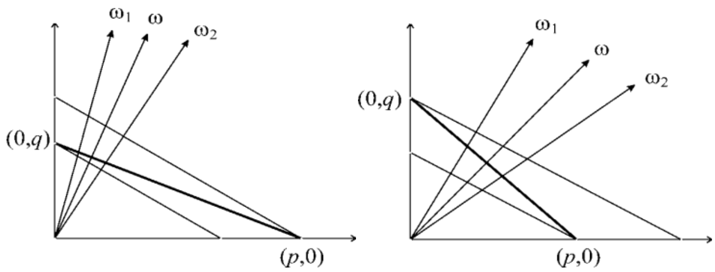


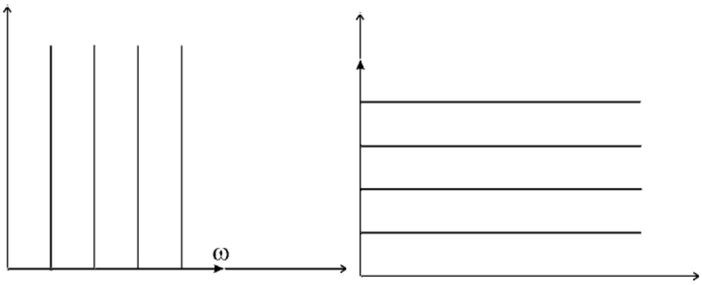
Figure 2

Two weight vectors. For the two points $(p, 0), (0, q) \in \mathbb{Z}^2$ one has $(p, 0) \prec_{\omega 1} (0, q)$, but $(p, 0) \prec_{\omega 2} (0, q)$. Hence m_1 and m_2 represent different term orders.



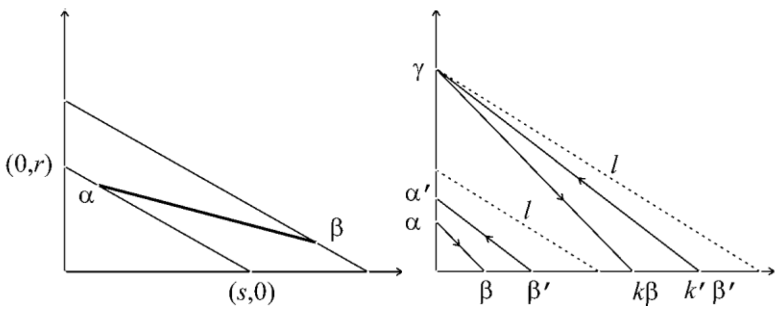
Proposition 3. *Any positive rational q defines exactly two different term orders.*

Proof. If we have $q = \frac{s}{r} \neq 0$ rational number, where $r, s \in \mathbb{Z}^2$ then the polynomial $f = x^s + y^r$ represents a tie between the terms. So here a" tiebreaking" order is needed. In two variables, it is a simple choice. We can choose it as *lex* with $x \prec y$ or *lex* with $y \prec x$.
Geometrically, in this case we have two points $(s, 0)$ and $(0, r)$ on the same line $x + qy = s$, and we have to compare them: either $(s, 0) \prec (0, r)$ *or* $(0, r) \prec (s, 0)$. We will use q^- to represent the term order defined by q with the tiebreaker of *lex* with $y \prec x$ and q^+ to represent the term order defined by q with the tiebreaker of *lex* with $x \prec y$. Obviously, q^+ and q^- are different term orders.
There are two exceptions. The first term order is described by the slope $0: m = 0, \omega = (1, 0)$ (we need only to consider the case 0^+). So that weight is given to the x component of the exponent vector, but no weight is given to the y component. Geometrically, the lines of this family are parallel to y-axis and contain infinitely many net points, and the order is uniquely determined by the condition $(0, 0) \prec (0, 1)$. This order is denoted by 0^+. It is actually x with $x \prec y$. The second one is the term order described by the slope ∞ (we need only to consider the case ∞^-): $m = \infty, \omega = (1, \infty) = (0, 1)$. Geometrically, the lines of this family are parallel to x-axis and contain infinitely many net points, and the order is uniquely determined by the condition $(0, 0) \prec (1, 0)$, this order is denoted by ∞^-. It is actually *lex* with $y \prec x$.



Example 1. *Let* $I = \langle xy^3 - x^2, x^3y^2 - y \rangle$ *and let us use* $\prec_{(1,2)}$ *as our term order*. We *can use Buchberger's algorithm to calculate a Gröbner basis for I. Let* $G = (g_1 = xy^3 - x^2, g_2 = x^3y^2 - y)$. Since $S(g_1, g_2) = -x^4 + y^2$ *and* $S(g_1, g_2)^G = -x^4 + y^2 \neq 0$, *we add* $S(g_1, g_2)^G$ *to* G *as new generator* $g_3 := -x^4 + y^2$. *Now set* $G = (g_1, g_2, g_3)$. *Computing S-polynomial we obtain:* $S(g_1, g_3) = y^5 - x^5$ *and* $S(g_1, g_3)^G \neq 0$. *We must add* $g_4 = y^5 - x^5$ *to our generating set, letting* $G(g_1, g_2, g_3, g_4)$. *Compute* $S(g_1, g_2)^G = S(g_1, g_3)^G = 0, S(g_2, g_3) = y^4 - xy$,
$S(g_2, g_3)^G \neq 0$, *We add* $g_5 := y^4 - xy$ *to G. Letting* $G = (g_1, g_2, g_3, g_4, g_5)$, *compute:* $S(g_2, g_3)^G = 0, S(g_1, g_4) = x^6 - x^2y^2, S(g_1, g_4)^G = 0, S(g_1, g_5) = S(g_1, g_5)^G = 0$, $S(g_2, g_4) = x^8 - y^4$ $, S(g_2, g_4)^G = 0$. $S(g_2, g_5) = x^4y - y^3, S(g_2, g_5)^G = 0$, *and* $S(g_3, g_4) = -y^7 + x^9$, $S(g_3, g_4)^G = 0$, $S(g_3, g_5) = -y^6 + x^5y, S(g_3, g_5)^G = 0, S(g_4, g_5) = -x^5 + xy^2, S(g_4, g_5)^G = 0$,
We see that $S(g_i, g_j)^G = 0$ *for all 1 ≤ i ≤ j ≤ 5, and it follows that G = (*g_1, g_2, g_3, g_4, g_5*) is a Gröbner basis for I with respect to* $\prec (1,2)$.

Theorem 1. *The set of all term orders on* $\mathbb{N}_0^2$ *is*
$\{0^+, \infty^-\} \cup \{q^+, q^-$*: q positive rational}* $\cup$ *{m: m positive irrational}.*
Proof. *For an arbitrary given term order* $\prec$ *define the set* $\Lambda_{\prec} = \{\frac{s}{r}: (s,0) \prec (0,r)\} \subset \mathbb{Q}$ take the least upper bound $\ell = sup\{\frac{s}{r}: (s,0) \prec (0,r)\}$ of the set $\Lambda_{\prec}$. According to previous constructions, this number determines another order $\prec_\ell$. Let us prove that this order is the same as the original one, i.e., that $\alpha \prec \beta \Leftrightarrow \alpha \prec_\ell \beta$.
First, analyze the two exceptional cases $\ell = 0$ *and* $\ell = \infty$. If $\ell = 0$, Then $s = 0$ and both orders are *lex* with $y \prec x$. if $\ell = \infty$ then for any $m = \frac{s}{r} \in \mathbb{Q}$ such that $(s, 0) \prec (0, r)$ there exists $m' = \frac{s'}{r'} > m, m \in \mathbb{Q}$, such that $(s', 0) \prec (0, r')$ both orders are *lex* with $x \prec y$.
Let now $\ell \in \mathbb{R}_+$ and take $\alpha = (x_1, y_1), \beta = (x_2, y_2) \in \mathbb{Z}_{\geq 0}^2$ such that $\alpha \prec \beta$. Compare the (Positive) slope $d = \frac{x_2 - x_1}{y_1 - y_2}$ with ℓ. There are two cases:
If $d > \ell$, then it is obvious that $\alpha \prec_\ell \beta$.
If $d \prec \ell$, take the rational slope q such that $d \prec q \prec \ell$ and the two corresponding endpoints α', β' such that $\beta' \prec_\ell \alpha'$. Suppose $\beta' \prec \alpha'$. Find the common multiple $\gamma = k\alpha = k'\alpha'$. Now, $k'\beta' \prec k'\alpha' = \gamma = k\alpha \prec k\beta$. Therefore $k'\beta' \prec k\beta$, which is impossible, therefore it must be the opposite.



This classification agrees with classification of Robbiano in [7] for the case $n = 2$.

2 Gröbner fan

If an ideal is given by finite set of generators, we cannot guess Gröbner fan of *I* from the Gröbner fans of generators. Rather, we should obtain the Gröbner basis *G* of *I*, for one monomial order $\prec$. The starting order $\prec$ and the members of the basis *G* determine one cone C_G of the Gröbner fan of I. Then, we cross the boundary of the cone C_G by choosing one of the neighboring orders $\prec_{new}$. There are two well-known ways to compute the corresponding cone. One, to apply basic algorithm for obtaining Gröbner basis, to *G* with the new monomial order. The other, to take the previous Gröbner basis *G*, to extract leading forms of polynomials in *G* with respect to boundary order $\prec_b$ and to compute (reduced) Gröbner basis *H* for the ideal they generate, with respect to the new target order $\prec_{new}$. If we denote by f^G the reduced form (remainder) of the polynomial *f* with respect to the starting order $\prec$ and with respect to *G*, then the Gröbner basis of *I* for the new order is $\{f - f^G | f \in H\}$. The use of the Newton polygon in the example follows Sturmfels [2].

Example 2. *Let us consider the ideal I =* $\langle xy^3 - x^2, x^3y^2 - y\rangle$. *We want to describe the Gröbner fan of I. The idea is to determine boundaries of its cones starting from the slope* 0^+ *and moving in positive direction along the arc in the first quadrant.*

1) The first step. Let $\prec_1$ be the lexicographical order. Its weight vector is (1, 0) with the slope 0. The corresponding matrix is

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For more details see [1], [4], [9]. We underline the leading terms is $f_1 \coloneqq xy^3 - \underline{x^2}$ and $f_2 \coloneqq \underline{x^3y^2} - y$ from *I*. Then $S_{1,2} = f_2 + xy^2 f_1 = -y + x^2y^5 = -y^5 f_1 + xy^8 - y = -y^5 f_1 + f_3$, where $f_3 \coloneqq \underline{xy^8} - y$. From the former, we eliminate f_2. Now, $S_{1,3} = y^8 f_1 + x f_3 = xy^{11} - xy = y^3 f_3 + y^4 - xy = y^3 f_3 + f_4$, where $f_4 \coloneqq y^4 - \underline{xy}$. Then $S_{1,4} = y f_1 - x f_4 = 0$ and $S_{3,4} = f_3 + y^7 f_4 = \underline{y^{11}} - y =: f_5$. From that we can eliminate f_3. We now consider only f_1, f_4, f_5. Further, $S_{1,5} = y^{11} f_1 + x^2 f_5 = xy^{14} - x^2 y = xy^3 f_5 + y f_1$, and $S_{4,5} = y^{10} f_4 + x f_5 = y^{14} - xy = y^3 f_5 + f_4$. *The basis* $\{f_1, f_4, f_5\} = \left\{xy^3 - \underline{x^2}, y^4 - \underline{xy}, \underline{y^{11}} - y\right\}$ is a Gröbner basis of *I* with respect to $\prec_1$. We take the reduced Gröbner basis of *I*, $G_1 = \left\{\underline{x^2} - y^6, \underline{xy} - y^4, \underline{y^{11}} - y\right\}$. From the Newton polygon of $y^6 - x^2$ we read $(2,0) - (0,6) = (2,-6) \perp (3,1)$. Similarly, using f_4 we have $(1,1) - (0,4) = (1,-3) \perp (3,1)$. Therefore, the vector $(3,1)$ spans one-dimensional cone, the border between two-dimensional cones in the Gröebner fan of *I*. then, $G_1 \coloneqq \mathbb{R}_{\geq 0}.(1,0) + \mathbb{R}_{\geq 0}.(3,1)$ is the cone in the Gröebner fan of *I* that corresponds to the basis G_1.

2) The second step. Assume that $\prec_2$ is the monomial order just above the slope $\frac{1}{3}^+$. Then, that corresponding matrix is

$$M_2 = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$$

Let $g_1 \coloneqq \underline{y^6} - x^2, g_2 \coloneqq \underline{y^4} - xy, g_3 \coloneqq \underline{y^{11}} - y$ from G_1. Then $S_{1,2} = g_1 - y^2 g_2 = \underline{xy^3} - x^2 =: g_4$. This eliminates g_1. $S_{2,3} = g_3 - y^7 g_2 = xy^8 - y = x(g_2 + xy)^2 - y = x(g_2 - 2xy) g_2 + g_5$, *where* $g_5 \coloneqq \underline{x^3y^2} - y$. We also eliminate g_3. $S_{2,4} = x g_2 - y g_4 = 0$, $S_{4,5} = x^2 g_4 - y g_5 = y^2 - \underline{x^4} =: g_6, S_{2,5} = x^3 g_2 - y^2 g_5 = -x^4 y + y^3 = y g_6$, $S_{2,6} = x^4 g_2 - y^4 g_6 = -x^5 y + y^6 = y^2 g_2 + xy g_6$, $S_{4,6} = x^3 g_4 + y^3 g_6 = -x^5 + y^5 = y g_2 + x g_6$, $S_{5,6} = x g_5 + y^2 g_6 = -xy + y^4 = g_2$. The set $G_2 = \left\{\underline{y^4} - xy, \underline{xy^3} - x^2, \underline{x^3y^2} - y, \underline{x^4} - y^2\right\}$ is the reduced Gröbner basis with respect to $\prec_2$. From the newton polygons of g_2 and g_4

we read $(0,4)-(1,1)=(1,3)-(2,0)=(-1,3)\perp(3,1)$. Similarly, using g_6 we have $(4,0)-(0,2)=(4,-2)\perp(1,2)$. Then , $G_2:=\mathbb{R}_{\geq 0}.(3,1)+\mathbb{R}_{\geq 0}.(1,2)$ is the cone in the Gröebner fan of *I* corresponding to the basis *G*2.

3) The third step. Assume that $\prec_3$ is the monomial order just above the slope 2^+. The corresponding matrix is

$$M_3=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$

We mark $h_1:=\underline{y^4}-xy, h_2:=\underline{xy^3}-x^2,\ h_3:=\underline{x^3y^2}-y, h_4:=\underline{y^2}-x^4$ from *G*2. Since the leading term of h_4 is y^4 and it divides all other leading terms, we will eliminate h_1,h_2 and h_3, by using syzygies.

$S_{2,4}=h_2-xyh_4=\underline{x^5y}-x^2=:h_5$,

$S_{3,4}=h_3-x^3h_4=\underline{x^7}-y=:h_6, S_{1,4}=h_1-y^2h_4=x^4y^2-xy=x^4h_4+x^8-xy=x^4h_4+xh_6+xh_6, S_{4,5}=x^5h_4-yh_5=-x^9+x^2y=-x^2h_6, S_{4,6}=x^7h_4-y^2h_6=-x^{11}+y^3=-x^4h_6+yh_4, S_{5,6}=x^2h_5-yh_6=-x^4+y^2=h_4$. The set $G_3:=\left\{\underline{y^2}-x^4,\underline{x^5y}-x^2,\underline{x^7}-y\right\}$ is the reduced Gröbner basis for $\prec_3$. From the newton polygons of h_4 we read $(4,0)-(0,2)=(4,-2)\perp(1,2)$. Also, using h_6 we have $(7,0)-(0,1)=(7,-1)\perp(1,7)$. Therefore, the cone corresponding to the basis G_3 is $G_3:=\mathbb{R}_{\geq 0}.(1,2)+\mathbb{R}_{\geq 0}.(1,7)$.

4) The fourth step. Assume that $\prec_4$ is the monomial order just above the slope 7^+. The corresponding matrix is

$$M_4=\begin{pmatrix}1 & 7\\ 0 & 1\end{pmatrix}$$

We mark $k_1:=\underline{y^2}-x^4, k_2:=\underline{x^5y}-x^2,\ k_3:=\underline{y}-x^7$ from *G*3. Since the leading term of k_3 is y and it divides all other leading terms, we will eliminate k_1 and k_2, by using syzygies.

$S_{1,3}=k_1-yk_3=-x^4+=x^7y=x^2k_2, S_{2,3}=k_2-x^5k_3=x^{12}-x^2=:k_4, S_{3,4}=yk_4=-x^{12}k_3=-x^2y+x^{19}=x^4k_4-x^2k_3$. The set $G_4:=\left\{\underline{y}-x^7,\underline{x^{12}}-x^2\right\}$ is the reduced Gröbner basis with respect to $\prec_4$. We have $(0,1)-(7,0)=(-7,1)\perp(1,7)$ from the newton polygons of k_3. Also, using k_4, we observe $(12,0)-(2,0)=(10,0)\perp(0,1)$. Then, $G_4:=\mathbb{R}_{\geq 0}.(1,7)+\mathbb{R}_{\geq 0}.(0,1)$ is the cone corresponding to the basis G_4.

3. Bivariate Gröbner fan algorithm

In the case $n=2$ a precise algorithm can be given.

INPUT: An ideal $I=\langle f_1,\ldots,f_S\rangle$.

OUTPUT: The Gröbner fan of the ideal, *GF*(*I*).

INTIALIZATION: $m=0, GF(I)=\emptyset$

WHILE: $m\geq 0, m\neq\infty$

Compute the reduced Gröbner basis of I with respect to $m^+, G_{m^+}=\{g_{m_1},g_{m_2},\ldots,g_{m_t}\}$ *calculate* $n,k\in\mathbb{Q}$, *where* $n\leq m<k$ *and n is the largest rational less than or equal to m and k is the smallest rational greater than m such that there exist* g_{m_i}, g_{m_j}*(not necessarily distinct) such that* $in_k(g_m)$ *and* $in_n(g_m)$ *are nonmonomial* . $GF(I):=GF(I)\cup\{the\ cone\ from\ the\ slop\ n\ to\ the\ slop\ k\}, m:=k.$

Conclusion

The most important ingredient in algorithm of polynomial rings is term order in this paper we showed the original geometric approach to classification of all possible orders on the ring of polynomials in two variables is given

References

[1] A. Ovchinnikov, A. Zobnin: Classification and application of monomial orderings and the properties of differential orderings.

[2] www.mayr.in.tum.de/konferenzen/CASC2002/CD/pdf/casc02.25.pdf.

[3] B. Sturmfels: *Gröbner bases and convex polytopes*. University Lecture Series, vol.8, AMS, Providence RI, 1996.

[4] D. Cox, J. Little, D. O'Shea: *Ideals, Varieties, and Algorithms*.Springer, New York,1997.

[5] D. Pritchard: *Walking Through The Gröbner Fan*. cit- seerx.ist.psu.edu/viewdoc.

[6] J. Freeke: Linking Gröbner Bases and Toric Varieties, master project. people.bath.ac.uk/ac886/students/JaquelineFreeke.pdf.

[7] K. Fukuda, N. Jensen and R. Thomas: *Computing Gröbner fans*, math.AC/0509544

[8] L. Robbiano: *Term orderings on the polynomial ring*. EUROCAL 85, vol.2 (Linz,1985), Lecture Notes in Comput. Sci , vol. 204, Springer, Berlin, 1985, pp. 513-517.

[9] S. Collart, M. Kalkbrener and D. Mall: *Converting Bases with the Gröbner Walk*. J.Symbolic Computation (1997) 24,465-469.
[10] S. Zeada: Polynomial division and Gröbner basis. The Teaching of Mathematics, 2013 (accepted).
[11] [10] W.W. Adams, P. Loustaunau: *An Introduction to Gröbner Bases*. Graduate Studies in Mathematics, vol. 3, AMS, Providence RI, 1994.