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A Generalized Deterministic Splitting Framework for the Modified Adomian Decomposition Method Applied to Nonlinear Partial Differential Equations

Aml Atiyah Mohammed[1*], Hanan Fuad Layyas[2]
[1,2]Department of Mathematics, Faculty of Sciences, University of Derna, Derna, Libya

إطار تجزئه حتمي وتعميمي لطريقة حل ادوميان المعدلة المطبقة على المعادلات التفاضلية الجزئية الغير خطية

امل عطية محمد[1*]، حنان فؤاد لياس[2]
[1،2]قسم الرياضيات، كلية العلوم، جامعة درنة، درنة، ليبيا

*Corresponding author: attiaamal040@gmail.com

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Abstract:
This paper presents a rigorous mathematical framework and a generalized theorem for the Modified Adomian Decomposition Method (MADM) applied to a broad class of nonlinear partial differential equations of the form $\frac{\partial u}{\partial t} + u^n \frac{\partial u}{\partial x} = X + X^n t^{n+1}$ under general initial conditions. Traditionally, the application of MADM relies on heuristic trial-and-error strategies to split the integrated source term into components f_1 and f_2 for self-canceling noise terms. To address this methodological limitation, we establish a deterministic splitting criterion and a general proof demonstrating that the proposed framework systematically ensures the vanishing of the first-order component $u_1 = 0$ for arbitrary positive integer exponents n. Consequently, this guarantees immediate series truncation at the first iteration, yielding exact closed-form solutions directly without relying on intuitive guesswork.

Key words: Modified Adomian Decomposition Method (MADM), Nonlinear Partial Differential Equations, Deterministic Splitting Criterion, Noise Terms, Exact Closed-Form Solutions.

الملخص:
تقدم هذه الورقة إطارا رياضياً صارم ونظرية معممة لطريقة حل ادوميان المعدلة (MADM) المطبقة على فئة من المعادلات التفاضلية الجزئية غير الخطية بالشكل $\frac{\partial u}{\partial t} + u^n \frac{\partial u}{\partial x} = X + X^n t^{n+1}$ تحت شروط أولية عامة تقليديا يعتمد تطبيق طريقة (MADM) على استراتيجيات التجربة والخطأ الاستدلالية لتقسيم الحد المصدر المتكامل الى مكونين f_1 and f_2 لحدود الضوضاء ذاتية للإلغاء. لمعالجة هذا القصور المنهجي ننشئ معيار تقسيم حتمي وبرهانا عاما يثبت ان الإطار المقترح يضمن بشكل منهجي تلاشي المكون من الرتبة الأولى $u_1 = 0$ للأسس الصحيحة الموجبة n ونتيجة لذلك يضمن هذا قطع المتسلسلة الفوري في التكرار الأول مما ينتج حلولا دقيقة مغلقة الشكل مباشرة دون الاعتماد على التخمين والحدس.

الكلمات المفتاحية: طريقة تحليل ادوميان المعدلة،(MADM) ، المعادلات التفاضلية الجزئية غير الخطية، الحلول الدقيقة مغلقة الشكل.

Introduction:

Nonlinear partial differential equations (PDEs) play a pivotal role in modeling complex physical and engineering phenomena. Because obtaining closed-form analytical solutions for such models using classical techniques is often challenging, semi-analytical and approximation methods have gained considerable attention. Among these, the Adomian Decomposition Method (ADM) has emerged as a powerful tool for handling nonlinear problems without the need for linearization or discretization.

Despite its effectiveness, the standard ADM can encounter slow convergence rates and computational complexities, particularly when dealing with nonhomogeneous equations accompanied by "noise terms" within the source functions. To overcome these limitations, Wazwaz introduced key modifications aimed at accelerating series convergence and effectively eliminating noise terms by strategically splitting the source terms.(1)

This paper applies the Modified Adomian Decomposition Method (MADM) via Wazwaz's modifications to a general class of nonlinear partial differential equations involving coupled non-linearities and spatio-temporal components. Specifically, we investigate the following nonlinear model:

$$\frac{\partial u}{\partial t} + u^{n}\frac{\partial u}{\partial x} = \mathcal{X} + \mathcal{X}^{n}t^{n+1} \quad (1)$$

where n is a positive integer representing the degree of non-linearity in the system. The primary objective of this work is to demonstrate the efficiency and rapid convergence of the modified scheme in recovering the exact analytical solution.

By properly formulating the initial and boundary conditions, we show that the proposed approach successfully eliminates noise terms, simplifies the iteration process, and ensures full consistency with the exact physical model. (2)

Mathematical Methodology:

Standard Adomian Decomposition Method (ADM):

The Adomian Decomposition Method (ADM) provides a direct analytical approach for solving nonlinear differential equations. Consider a general nonlinear partial differential equation expressed in operator **form:**

$$L_t u + Ru + Nu = g(x,t) \quad (2)$$

where $L_t = \frac{\partial}{\partial t}$ is the highest-order linear differential operator (assumed to be easily invertible), R is the remaining linear differential operator, Nu represents the non-linear terms, and $g(x,t)$ is the source term.

Applying the inverse operator $L_t^{-1} = \int_0^t (.)\,dt$ to both sides and incorporating the initial condition $u(x,0) = f(x)$

Yields the integral equation:

$$u(x.t) = f(x) + L_t^{-1}[g(x,t)] - L_t^{-1}[Ru(x,t)] - L_t^{-1}[Nu(x,t)] \quad (3)$$

According to the standard ADM, the unknown function $u(x,t)$ is represented as an infinite decomposition series:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) \quad (4)$$

where A_m are the Adomian polynomials representing the nonlinear term N_u , defined as:

$$N_u = \sum_{m=0}^{\infty} A_m \quad (5)$$

The Adomian polynomials A_m are calculated using **the general formula:**

$$A_m = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, m = 0,1,2,3\ldots\ldots \quad (6)$$

Substituting the decomposition series of $u(x,t)$ and Nu into **the integral equation yields:**

$$\sum_{m=0}^{\infty} u_m(x,t) = f(x) + L_T^{-1}[g(x,t)] - L_t^{-1}\left[R\left(\sum_{m=0}^{\infty} u_m(x,t)\right)\right] - L_t^{-1}\left[\sum_{m=0}^{\infty} A_m\right] \quad (7)$$

Based on the standard ADM recursive scheme, the component $u_m(x,t)$ are determined **as follows:**

$$u_0(x,t) = f(x) + L_t^{-1}[g(x,t)] \quad (8)$$

$$u_{m+1}(x,t) = -L_t^{-1}\left[R\left(u_m(x,t)\right)\right] - L_t^{-1}[A_m] \quad , m \geq 0 \quad (9)$$

The k-term approximate solution is given by:

$$\emptyset_k(x,t) = \sum_{m=0}^{k-1} u_m(x,t) \quad (10)$$

and the exact solution is obtained as, k⟶ ∞ Modified Adomian Decomposition Method (MADM) via Wazwaz's Modifications To improve the computational efficiency of the standard ADM and accelerate series convergence, Wazwaz introduced effective modifications, particularly for nonhomogeneous equations containing source terms $g(x,t)$ In the modified scheme, the source term $g(x,t)$ is decomposed into two parts:

$$g(x,t) = g_1(x,t) + g_2(x,t) \qquad (11)$$

Instead of assigning the entire source term to the zeroth-order component$u_0(x,t)$, the modified recursive relation assigns only part of the source term along with the initial condition to$u_1(x.t)$, while the remaining part $g_2(x,t)$ is passed to$u_1(x,t)$. Thus, the recurrence relation becomes:

$$u_0(x,t) = f(x,t) + L_t^{-1}[g_1(x,t)] \qquad (12)$$

$$u_1(x,t) = L_t^{-1}[g_2(x,t)] - L_t^{-1}[R(u_0(x,t))] - L_t^{-1}[A_0] \qquad (13)$$

$$u_{m+1}(x,t) = -L_t^{-1}(R(u_m(x,t) - L_t^{-1}[A_m], \quad m \geq 1 \qquad (14)$$

This strategic splitting allows for the early cancellation of identical self-canceling terms (referred to as "noise terms") between$u_0(x,t)$ and$u_1(x,t)$, leading to the exact solution in a very few iterations.

To enhance the computational efficiency of the Modified Adomian Decomposition Method (MADM) and ensure rapid convergence, we introduce a structured splitting strategy for the source term $g(x,t)$Instead of an arbitrary decomposition$g(x,t) = g_1(x,t) + g_2(x,t)$, the proposed criterion selects the lowest-order temporal component of the source function to construct the zeroth-order approximation$u_0(x,t)$.(3)

By assigning the minimal power term of t to $g_1(x,t)$ alongside the initial condition f(x), we accelerate the identification and immediate cancellation of self-canceling "noise terms" appearing between $u_0(x,t)$ and$u_1(x,t)$ This structural choice simplifies the evaluation of Adomian polynomials A_m and yields the exact solution in very few iterations .(4)

1. **Step-by-Step Implementation for Model (1):** We demonstrate the implementation of the proposed method on the general nonlinear partial differential equation given by Eq 1-Operator Form & Source Splitting Rewrite the equation using the inverse time operator$L_t^{-1}\int_0^t(.)\,dt$ and apply Wazwaz's source term splitting ($f_1 = x, f_2 = x^n t^{n+1}$)

$$u(x,t) = u(x,0) + L_t^{-1}(f_1) + \left[L_t^{-1}(f_2) - L_t^{-1}(u^n \frac{\partial u}{\partial x})\right]$$

2. **Zeroth Component:** With the initial condition $u(x,0) = 0$

$$u_0(x,t) = \int_0^t x\,dt = xt$$

3. **Modified Iterative Scheme (Wazwaz):** Using Wazwaz's modification to eliminate noise terms directly

$$u_1(x,t) = L_t^{-1}(f_2) - L_t^{-1}(A_0)$$

Where $A_0 = u_0^n \frac{\partial u_0}{\partial x}$ Is the Adomian polynomial for the nonlinear term (5)

4. **Evaluation & Conclusion:** Substitute $u_0 = xt \;\; into \;\; A_0 = (xt)^n \frac{\partial}{\partial x}(xt) = x^n t^{n+1}$. Evaluate the integrals:

$$L_t^{-1}(f_2) = \int_0^t x^n t^{n+1}\,dt = x^n \frac{t^{n+2}}{n+2}$$

$$L_t^{-1}(A_0) = \int_0^t x^n t^{n+1}\,dt = x^n \frac{t^{n+2}}{n+2}$$

$$u_1(x,t) = x^n \frac{t^{n+2}}{n+2} - x^n \frac{t^{n+2}}{n+2} = 0$$

Since $u_1 = 0$ all subsequent components vanish ($u_m = 0 \; for \; m \geq 1$)

Final Exact solution:

$$u(x,t) = u_0(x,t) = xt \quad if \quad u(x,0) = 0$$

But:

$$u(x,0) = \hbar$$

Then final exact solution:

$$u(x,t) = u_0(x,t) = \hbar + xt$$

The primary objectives of this paper are outlined as follow:

1. To establish a deterministic, non-heuristic splitting framework for the integrated source term in the Modified Adomian Decomposition Method (MADM) applied to nonlinear PDEs of the form (1).
2. To rigorously prove a generalized theorem ensuring the immediate vanishing of the first-order component $u_1 = 0$for any arbitrary positive integer exponent n.

3. To eliminate the trial-and-error limitations inherent in traditional noise term splitting criteria, thereby guaranteeing exact closed-form solutions in a single iterative step.
4. To validate the proposed theoretical framework through diverse numerical applications, including polynomial and transcendental test cases.

Application the Casa of n=1:

To demonstrate the application of the proposed framework, we first consider the non-linear partial differential equation withe n=1:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{x} = x + xt^2 \, , u(x,0) = \hbar$$

Applying the inverse integration operator $L_t^{-1}(.) = \int_0^t dt$ to the source term gives the integrated function:

$$f(x,t) = \int_0^t (x + xt^2)\, dt = xt + \frac{xt^3}{3}$$

According to the deterministic splitting rule,$f(x,t)$ is decomposed into $F_1(x,t) = xt$. And $f_2(x,t) = \frac{xt^3}{3}$ The zeroth-order component is set as $u_0(x,t) = f_1(x,t) = xt$.

The first Adomian polynomial A_0 for the nonlinear term $N(u) = u\frac{\partial u}{\partial x}$ is computed using u_0

$$A_0 = u_0\frac{\partial u_0}{\partial x} = (xt).\frac{\partial}{\partial x}(xt) = xt^2$$

Substituting $f_2(x,t)\ and\ A_0$ into **the first-order relation yields:**

$$u_1(x,t) = f_2(x,t) - L_t^{-1}(A_0) = \frac{xt^3}{3} - \int_0^t xt^2 = \frac{xt^3}{3} - \frac{xt^3}{3} = 0$$

$$u(x,t) = u(x,0) + u_0(x,t)$$

$$u(x,t) = \hbar + xt$$

Application to a cubic nonlinear partial differential equation:

Consider the non- linear partial equation with a cubic non-linear term (n=3):

$$\frac{\partial u}{\partial t} + u^3\frac{\partial u}{\partial t} = x + x^3t^4, u(x,0) = 0$$

Applying the inverse integration operator $L_t^{-1}(.) = \int_0^t (.)\, dt$

To the source term gives the integrated function:

$$f(x,t) = \int_0^t (x + x^3t^4)dt = xt + \frac{x^3t^5}{5}$$

According to the proposed deterministic splitting rule,$f(x,t)$ is decomposed into$f_1(x,t) = xt$ and $f_2(x,t) = \frac{x^3t^5}{5}$ The zeroth-order component is set as:

$$u_0(x,t) = f_1(x,t) = xt$$

The first Adomian polynomial for the nonlinear operator$N(u) = u^3\frac{\partial u}{\partial x}$ is evaluated using u_0:

$$A_0 = (u_0)^3\frac{\partial u_0}{\partial x} = (xt)^3.\frac{\partial}{\partial x}\ (xt) = x^3t^4.$$

Substituting $f_2(x,t)$and A_0into the first-order relation yields:

$$u_1(x,t) = f_2(x,t) - L_t^{-1}(A_0) = \frac{x^3t^5}{5} - \int_0^t x^3t^4dt = \frac{x^3t^5}{5} - \frac{x^3t^5}{5} = 0$$

Since$u_1(x,t) = 0$, all subsequent higher-order components vanish $(u_m = 0, m \geq 1)$. Thus, the series terminates immediately at the initial step, yielding the exact closed-form solution:

$$u(x,\mathrm{t}) = u_0(x.\mathrm{t}) = x\mathrm{t}$$

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