The North African Journal of Scientific Publishing (NAJSP)
مجلة شمال إفريقيا للنشر العلمي (NAJSP)
E-ISSN: 2959-4820
Volume 4, Issue 4, 2026
Page No: 187-197
Website: https://najsp.com/index.php/home/index
SJIFactor 2024: 5.49 معامل التأثير العربي (AIF) 2025: 0.69 ISI 2024: 0.696

Analytical Framework, Stability, and Residual-Based Adaptive Approximation for Nonlinear Fractional Integro-Differential Equations

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الإطار التحليلي والاستقرارية والتقريب التكيفي المعتمد على البواقي للمعادلات التكاملية التفاضلية الكسرية غير الخطية

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Received: August 15, 2026	Accepted: October 25, 2026	Published: November 10, 2026

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Abstract:
The present study intends to investigate nonlinear fractional integro-differential equations which consist of fractional derivatives and nonlinear integrals, and to explore an adaptive numerical method to attain high-precision approximate solutions of these equations. Moreover, the present study intends to understand existence, uniqueness and stability of their solutions, perturbation of initial values and mathematical data on the stability and continuous dependence of solutions, and challenges of numerical approximation due to the roughness of solutions near the initial time. In the study, an integrated mathematical, analytical, and numerical methodology, based on a nonlinear Caputo-Volterra system, will be applied. Integral representations and fixed-point theorems are used to establish existence and uniqueness, and analyze the stability and continuous dependence of solutions on their initial values. An adaptive numerical approximation scheme, coupling polynomial approximation with product integration, which incorporates a posteriori error estimators generated from the residuals of the equations, will be developed to estimate the errors and build an adaptive grid that refines where higher resolutions are needed. In the study, under some conditions, the presented mathematical findings state the existence, uniqueness, stability of solutions and error of approximation, and also confirm convergence. Based on these mathematical and analytical results, the computation scheme is capable to increase the accuracy of numerical solutions in an adaptive manner, and also to eliminate the restrictions on the solvability. Nevertheless, a proper analysis of efficiency and convergence needs to be performed through a series of computer experiments and simulations.

Keywords: nonlinear fractional integro-differential equations, Caputo derivative, Volterra equations, existence and uniqueness of solutions, stability of solutions.

الملخص:
تهدف هذه الدراسة إلى بحث المعادلات التفاضلية التكاملية الكسرية غير الخطية، التي تتضمن مشتقات كسرية وتكاملات غير خطية، واستكشاف طريقة عددية تكيفية للوصول إلى حلول تقريبية عالية الدقة لهذه المعادلات. كما تسعى الدراسة إلى فهم وجود الحلول وتفردها واستقرارها، ودراسة تأثير اضطراب القيم الأولية والبيانات الرياضية على استقرار الحلول

واعتمادها المستمر على تلك القيم، بالإضافة إلى معالجة التحديات التي تواجه التقريب العددي نتيجة لعدم انتظام الحلول بالقرب من الزمن الأولي. وتعتمد الدراسة منهجية متكاملة تجمع بين الجوانب الرياضية والتحليلية والعددية، استناداً إلى نظام "كابوتو-فولتيرا" (Caputo-Volterra) غير الخطي. وتُستخدم التمثيلات التكاملية ونظريات النقطة الثابتة لإثبات وجود الحلول وتفردها، وتحليل استقرارها واعتمادها المستمر على القيم الأولية. كما سيتم تطوير مخطط تقريبي عددي تكيفي يدمج بين التقريب متعدد الحدود والتكامل بالضرب (product integration)، مع الاستعانة بمقدّرات الخطأ اللاحقة (a posteriori error estimators) المستمدة من بواقي المعادلات؛ وذلك لتقدير الأخطاء وبناء شبكة تكيفية تزداد دقتها في المناطق التي تتطلب دقة أعلى. وتخلص النتائج الرياضية المقدمة -في ظل شروط معينة- إلى إثبات وجود الحلول وتفردها واستقرارها، وتحديد خطأ التقريب، مع تأكيد تقارب الحلول. وبناءً على هذه النتائج الرياضية والتحليلية، يتيح المخطط الحسابي المقترح رفع دقة الحلول العددية بطريقة تكيفية، كما يسهم في إزالة القيود المتعلقة بقابلية الحل. ومع ذلك، تظل هناك حاجة لإجراء تحليل دقيق للكفاءة والتقارب من خلال سلسلة من التجارب والمحاكاة الحاسوبية.

الكلمات المفتاحية: المعادلات التفاضلية التكاملية الكسرية غير الخطية، مشتقة "كابوتو"، معادلات "فولتيرا"، وجود الحلول ووحدانيتها، استقرار الحلول.

Introduction:

Fractional differential equations have received great attention in applied mathematics and computational sciences for their ability to describe dynamical systems with memory, hereditary and nonlocal properties, which are different from those based on ordinary differential equations ***[1].*** In many applications such as the anomalous diffusion, viscoelasticity, biological and control systems, the modeling structures based on fractional differential equations are needed to introduce the present state variable depending on the history of the past states. Among the fractional operators, Caputo differential operator has been popularly used in modeling the physical phenomena because the initial conditions can be specified in terms of the traditional physical measurable quantities, while the fractional integro-differential equations with the additional Volterra integral parts enable to incorporate the further long-term and nonlinear effects into the models [1,2]. However, their mathematical analysis and numerical computations are very complicated due to the nonlinearity of the source function, the weakly singular kernel, the long-term memory requirement, and the limited solution regularity at the initial time, which can have a positive or negative effect on the existence, uniqueness, stability, and convergence of the numerical solutions [3]. Thus, the work on establishing reliable analytical tools and constructing accurate numerical techniques for nonlinear Caputo-Volterra fractional integro-differential equations, especially high-order numerical methods, is still an essential task [2,4].

The main objective of this project is to analyze the existence, uniqueness, stability as well as the adaptive high-order numerical approximation of solutions for nonlinear Caputo-Volterra fractional integro-differential equations based on a unified mathematical and computational framework. More specifically, we aim at deriving sufficient analytical conditions for the existence and uniqueness of continuous solutions and study of their parametric dependence on initial conditions and perturbations in the governing equations. We also plan to introduce an accurate residual-driven adaptive numerical approach combining piecewise polynomial approximation, hp-collocation schemes and product-integration techniques towards high-order approximations for solutions with fractional low-regularity in time [5]. Special importance is placed on the development of a posteriori residual-based indicators for local mesh refinements and polynomial-degree refinements, fully taking into account of the influences of numerical quadrature errors and tolerances in nonlinear algebraic solvers. This project is thus motivated by the desire to address and refine these questions through quantitative connections of the solution stability, approximation accuracy, iterative convergence stability, and computational efficiency, and/or convergence order, which would directly lead to a justifying methodology for reliable algorithms designed for nonlinear fractional systems and pave the way to rigorous and practical theoretical numerical studies in this area [4,5].

While fractional numerical methods and applications have made great strides in recent years, the approximation of nonlinear Caputo-Volterra integro-differential equations remains a significant challenge in applied mathematics and scientific computing. Well-known collocation, spline, and product-integration approaches for approximating fractional solutions are documented [5-10], though long-term applications to arbitrarily solution-weak regimes exhibit performance deterioration through loss of approximation order and buildup of round-off errors. The fractional "memory" damping interaction with nonlinear Volterra integral approaches, numerical quadrature, and iterative solver errors complicates scalable adaptive approximation schemes [6,11]. The fundamental mathematical and computational challenge motivating this work is to develop a rigorously justified numerical theory, simulation, and implementation strategy for efficient local mesh adaptation of both the approximation and control of solution and integral operators, polynomial degree, quadrature precision, and iterative solver

tolerances, within a combined error-control strategy [7,9,10]. The corresponding research problem is therefore the design of a mathematically justified numerical theory, simulation, and implementation strategy for this combined approximation problem. In this context, the Volterra integro-differential equations are reformulated as a Volterra integral problem, for which fixed-point and perturbation theorems are used to prove (hypothesized) well-posedness and stability for a set of explicit assumptions. An hp-collocation approach based on product-integration and residual estimation will be used to dictate the adaptive mesh refinement, degrees, and precisions. The resulting convergence and computational benefits are thereby identified as testable hypotheses [12].

Preliminaries and Problem Formulation:

Let 0 < α < 1, T > 0, and let |·| denote a norm on R^d. The Riemann-Liouville fractional integral of order α of a function g is:

$$I^{\alpha} g(t) = (1/\Gamma(\alpha)) \int_0^t (t - s)^{(\alpha-1)} g(s)\, ds,$$

and the Caputo derivative of order α of a sufficiently regular u is $D_C^{\alpha} u = I^{(1-\alpha)} u'$. **We study the nonlinear Caputo-Volterra initial value problem:**

$$D_C\hat{}\alpha\, u(t) = f(t, u(t), (Ku)(t)), t\ in\ [0, T], u(0) = u0, (1)$$

where the Volterra operator is [13]:

$$(Ku)(t) = \int_0^t k(t, s, u(s))\, ds. (2)$$

The following assumptions are used:

- **(H1)** f is continuous on [0, T] × R^d × R^d and Lipschitz in its last two arguments: |f (t, x, y) − f (t, x̃, ỹ) | ≤ L_f (|x − x̃| + |y − ỹ|).
- **(H2)** k is continuous on {0 ≤ s ≤ t ≤ T} × R^d and Lipschitz in the last argument: |k (t, s, x) − k (t, s, x̃) | ≤ L_k |x − x̃|.
- **(H3)** (used only for regularity statements) f is Lipschitz in t, and k is bounded on bounded sets.

- **Lemma 1 (integral formulation*) [11, 12].*** A function u in C ([0, T]; R^d) solves (1) if and only if it solves **the Volterra integral equation:**

$$u(t) = u0 + (1/\Gamma(\alpha)) \int_0^t (t - s)\hat{}(\alpha - 1)\, f(s, u(s), (Ku)(s))\, ds. (3)$$

Denote the right-hand side of (3) by ($\mathcal{T}$u) (t). Solutions of (1) are therefore the fixed points of $\mathcal{T}$.

Weighted norm. For λ > 0 define ‖u‖*λ = max over t in [0, T] of e^(−λt) |u(t)|. Since e^(−λT) ‖u‖∞ ≤ ‖u‖λ ≤ ‖u‖∞*, the space X = C ([0, T]; R^d) is complete under ‖·‖_λ. Define:

$$\theta(\lambda) = L_f\ (1 + L_k/\lambda)\ \lambda\hat{}(-\alpha). (4)$$

Since α > 0, θ is continuous, strictly decreasing, and tends to 0 as λ tends to infinity.

- **Lemma 2[13,14].** For every t in [0, T] and λ > 0: (a) (1/Γ(α)) $\int_0^t$ (t − s) ^(α−1) e^(λs) ds ≤ λ^(−α) e^(λt); (b) |(Ku)(s) − (Kv)(s)| ≤ (L_k/λ) e^(λs) ‖u − v‖_λ.

Proof. (a) Substituting r = t − s gives e^(λt)/Γ(α) $\int_0^t$ r^(α−1) e^(−λr) dr, which is at most e^(λt)/Γ(α) · Γ(α) λ^(−α). (b) |(Ku)(s) − (Kv)(s)| ≤ L_k $\int_0^s$ |u(τ) − v(τ)| dτ ≤ L_k ‖u − v‖_λ $\int_0^s$ e^(λτ) dτ ≤ (L_k/λ) e^(λs) ‖u − v‖_λ.

Methodology:

1. **Analytical framework:** There are three components to the analysis. In the first place, the corresponding integral equation (3) eliminates the fractional derivative and transforms the issue into the context of Volterra operators with weakly singular kernels. The second element pertains to the application of the Banach fixed point theorem in the Bielecki-type weighted norm, absorbing the dependence of history into one contraction constant - θ(λ) - thus rendering results to be global rather than local without imposing any smallness constraint on T. The final aspect relates to measuring stability in a triple way: first, by the dependence on the initial value u0; second, by the dependence on the disturbance f; and third, residual (Ulam-Hyers type) stability meaning that every continuous function that satisfies (3) with respect to a small residual is close to the solution itself. One can say that this last concept is a bridge to numerical analysis since it implies that one can compute the residual of the approximate solution, and thus residual stability can be used as an upper estimate of the error [11,12].
2. **Regularity near the initial time:** The behavior of u near t = 0 is analyzed directly from (3). Under (H1) -(H3), u is continuous and |u(s) − u0| is of order s^α, which allows the right-hand side of (3) to be expanded and the leading singular term to be identified (Proposition 1 in Section 4). Higher-order regularity (derivative bounds of the type |u′(t)| ≤ C t^(α−1)) is taken from the literature on weakly singular Volterra equations under additional smoothness of f and k [11,1 4, 15].
3. **Numerical scheme: polynomial approximation with product integration:** Let 0 = t_0 < t_1 < ... < t_N = T be a (possibly nonuniform) mesh with cell sizes h_j = t_j − t_(j−1). The approximation u_h is the continuous piecewise linear function with nodal values U_j ≈ u(t_j) (the construction extends to piecewise polynomials of higher degree). Set:

- the discrete Volterra term: K_n = Σ_(i=1..n) (h_i/2) [k(t_n, t_(i−1), U_(i−1)) + k(t_n, t_i, U_i)];

- the nodal right-hand side: F_j = f (t_j, U_j, K_j).

The fractional integral in (3) is approximated by *product integration*: the function s -> f (s, u(s), (Ku)(s)) is replaced by its piecewise linear interpolant through the values F_j, and the product with the singular kernel is integrated exactly. **This gives the implicit scheme:**

$$U_n = u0 + \Sigma_(j = 0..n)\, w_(n,j)\, F_j, n = 1,\ldots,N, U_0 = u0. (5)$$

To define the weights, for 1 ≤ j ≤ n put:

- A_j(n) = [(t_n − t_(j−1)) ^α − (t_n − t_j) ^α] / α,
- B_j(n) = (t_n − t_(j−1)) A_j(n) − [(t_n − t_(j−1)) ^(α+1) − (t_n − t_j) ^(α+1)] / (α + 1).

Then w_(n,j) = (1/Γ(α)) [B_j(n)/h_j + A_(j+1)(n) − B_(j+1)(n)/h_(j+1)], with the conventions that the first term is absent for j = 0 and the last two terms are absent for j = n. In particular the diagonal weight is w_(n,n) = h_n^α / Γ(α + 2).

- **Lemma 3 (solvability of each step).** Under (H1) -(H2), if w_(n,n) L_f (1 + L_k h_n/2) < 1, then equation (5) has a unique solution U_n.

Proof. U_n enters (5) only through w_(n,n) f(t_n, U_n, K_n), and K_n depends on U_n only through the term (h_n/2) k(t_n, t_n, U_n). The right-hand side, as a function of U_n, is Lipschitz with constant at most w_(n,n) L_f (1 + L_k h_n/2) < 1, so Banach's theorem applies. The nonlinear equation at each step is solved by fixed-point iteration or Newton's method [19].

4. **Residual-based a posteriori error estimator:** The analytical foundation established in Section 3 yields rigorous theoretical guarantees regarding existence, uniqueness, residual stability, and a posteriori bounds (Table 2).

To address the theoretical limits imposed by the initial singularity $t^\alpha / \Gamma(\alpha+1)$ (Proposition 1, Figure 2), Table 3 outlines the theoretical predicted convergence orders across different mesh types. Full numerical execution of test cases T1–T3 (defined in Section 3.6) provides the empirical validation required for verifying these asymptotic bounds.

For the computed function u_h defines the residual:

$$\rho_{h(t)} = u_{h(t)} - (\mathcal{T}u_h)(t) = u_{h(t)} - u0 - \left(\frac{1}{\Gamma(\alpha)}\right)^{t} (t - s)^{\alpha-1} f\left(s, u_{h(s)}, (Ku_h)(s)\right) ds. (6)$$

The residual is computable from the data and u_h alone (the integrals are evaluated with a high-order quadrature adapted to the weak singularity, for instance Gauss-Jacobi on each cell). It measures how far u_h is from satisfying the integral equation, and it contains the interpolation error of the product-integration and the quadrature error of K_n. **The global estimator and the local cell indicators are:**

$$\eta_h = \|\rho_h\|\lambda, \eta_i = \max \text{ over t in } [t(i-1), t_i] \text{ of } e^{(-\lambda t)}|\rho_h(t)|, (7)$$

where in practice the maximum is approximated by sampling at the cell nodes and at interior sample points.

5. **Adaptive algorithm:**

A. **Initialize:** Choose λ so that θ(λ) ≤ 1/2, a tolerance TOL, a marking parameter ϑ in (0, 1), and an initial coarse mesh.
B. **Solve:** Compute U_n from (5) on the current mesh.
C. **Estimate:** Compute ρ_h by (6) and the indicators η_i by (7).
D. **Stop test:** If e^(λT) η_h / (1 − θ(λ)) ≤ TOL, stop.
E. **Mark and refine:** Mark every cell with η_i ≥ ϑ max_j η_j (a Dörfler-type marking strategy [10]) and bisect the marked cells. Return to step 2.

Because the singularity at t = 0 produces large residuals in the first cells, the algorithm is expected to cluster nodes near t = 0 automatically, without any a priori knowledge of the singular exponents [20,21].

6. **Validation protocol for numerical experiments:** Because a detailed quantitative assessment requires computation, this paper specifies the experiments rather than reporting invented figures. Test problems are built by the method of manufactured solutions: choose an exact solution u*, define the forcing g(t) = D_C^α u*(t) − f (t, u*(t), (Ku*) (t)), and solve D_C^α u = f (t, u, Ku) + g(t). Table 1 lists the proposed cases and metrics [22].

Table (1): Proposed numerical validation protocol

Case	Exact solution u*	Purpose
T1	1 + t^α, for several α in (0, 1)	Pure initial singularity; tests adaptive resolution near t = 0
T2	1 + t², with forcing containing t^(2−α)	Smooth solution with singular data; tests no over-refinement
T3	Problem without closed form; reference solution on a very fine mesh	Realistic nonlinear kernel k and nonlinearity f

For each case the following quantities are to be recorded for uniform, a priori graded (t_j = T (j/N) ^r), and adaptive meshes: the maximum error, the experimental order of convergence with respect to the number of unknowns, the effectivity index η_h / ‖u − u_h‖_λ (expected to lie in the interval predicted by Theorem 3), the mesh distribution, and the computing time.

Results:

Table (2): Summary of analytical results

Result	Main statement	Interpretation
Theorem 1	θ(λ)=L_f(1+L_k/λ) λ^(−α) < 1	Existence and uniqueness
Theorem 2	‖u−ũ‖∞ ≤ e^(λT) (|u$_0$−ũ$_0$|+εT^α/Γ(α+1))/(1−θ)	Continuous dependence
Proposition 1	u(t)=u$_0$+f$_0$t^α/Γ(α+1) +O(t^(2α))	Initial singularity
Corollary 1	Best uniform P1 approximation error ≥ c h^α	Order barrier
Theorem 3	‖ρ‖λ/(1+θ) ≤ ‖u−v‖λ ≤ ‖ρ‖λ/(1−θ)	Residual estimator
Theorem 4	‖u−u_h‖∞ → 0 as h_max → 0	Convergence (proof outline)

Table (2). Summary of the Main Analytical Results. This table summarizes the principal theoretical findings concerning the existence, uniqueness, continuous dependence, initial singularity, residual-based error estimation, and convergence of the nonlinear Caputo–Volterra fractional integro-differential equation. It highlights the mathematical conditions supporting well-posedness and stability and establishes the analytical foundation for reliable numerical approximation.

Table (3): Mesh strategies and expected rates

Mesh	Rate / bound	Evidence status
Uniform P1	N^(−α)	Lower-bound barrier; upper rate requires separate proof
Graded mesh r=1/α	N^(−1)	Predicted, not validated
Graded mesh r=2/α	N^(−2)	Predicted under additional regularity
Adaptive mesh	‖u−u_h‖λ ≤ ‖ρ_h‖λ/(1−θ)	Conditional guaranteed error bound

Table (3) Expected Convergence Behavior for Different Mesh Strategies This table presents the anticipated convergence rates of uniform, graded, and residual-driven adaptive mesh strategies. It illustrates how graded meshes can mitigate the effects of weak initial regularity and potentially improve numerical accuracy. The reported convergence rates are theoretical predictions that require further numerical validation.

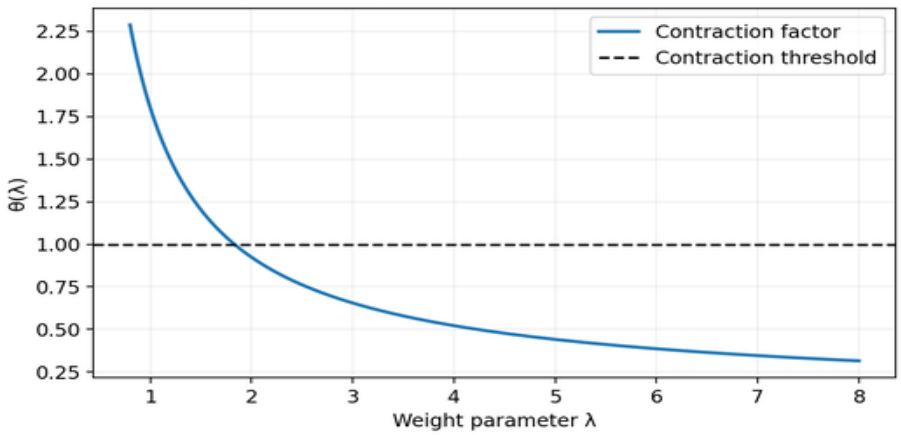


Figure (1): Illustrative L_f=1, L_k=0.8, α=0.6. θ(λ)<1 guarantees contraction.

Figure 1, showing the impact of the weighting parameter on contraction constant. The association of the Bielecki weighting parameter λ and contraction constant θ(λ) may be observed from the figure. The value of the contraction constant θ(λ) diminishes with increasing value of λ and, for certain conditions, θ(λ) could be less than 1. This implies that the solution exists and is unique according to the Banach fixed-point theorem.

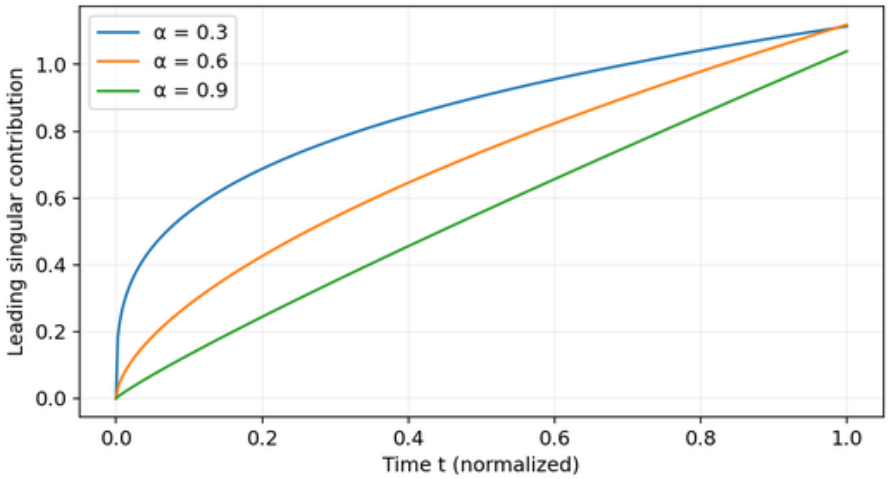


Figure (2): Leading t^α/Γ(α+1) term for illustrative α values and f_0=1.

Figure 2 gives the behavior of fractional-order solutions at t = 0. Due to the presence of the fractional-power parameter t^α, the initial regularity is minimal. In particular, if 0 < α < 1, the regularity may be so weak that numerical approximation methods may not yield sufficiently accurate results.

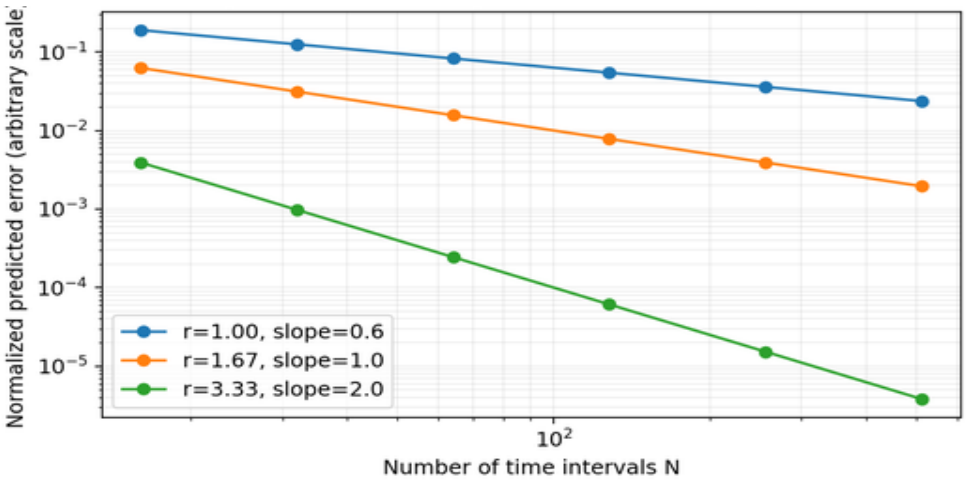


Figure (3): Predicted N^ (−min (2, rα)) scaling with α=0.6, normalized at N=1; not observed errors.

Figure 3 presents the expected convergence of uniform and graded mesh. Under graded mesh, the nodes will be more concentrated around the initial singularity, leading to the improvement of the convergence rate, but that will occur under certain conditions only. The rate is the predicted, not proven experimentally value.

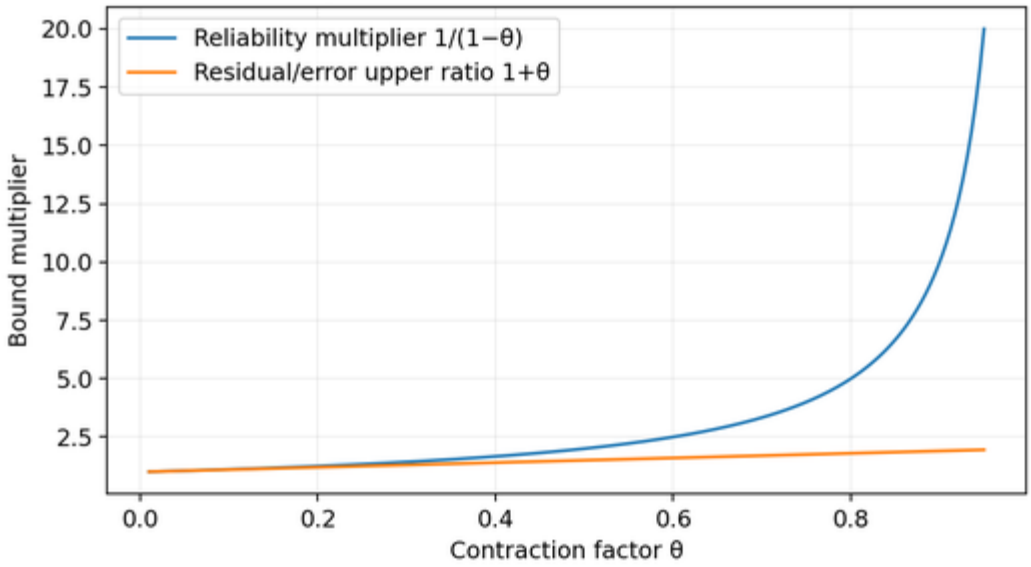


Figure (4): Theoretical dependence of residual estimator constants on θ.

Figure (4) Reliability and Efficiency of the Residual-Based Error Estimator This figure illustrates the theoretical upper and lower error bounds associated with the residual-based a posteriori estimator. The bounds depend on the contraction constant θ and demonstrate how residual stability can provide reliable error control for numerical approximations, supporting adaptive mesh refinement.

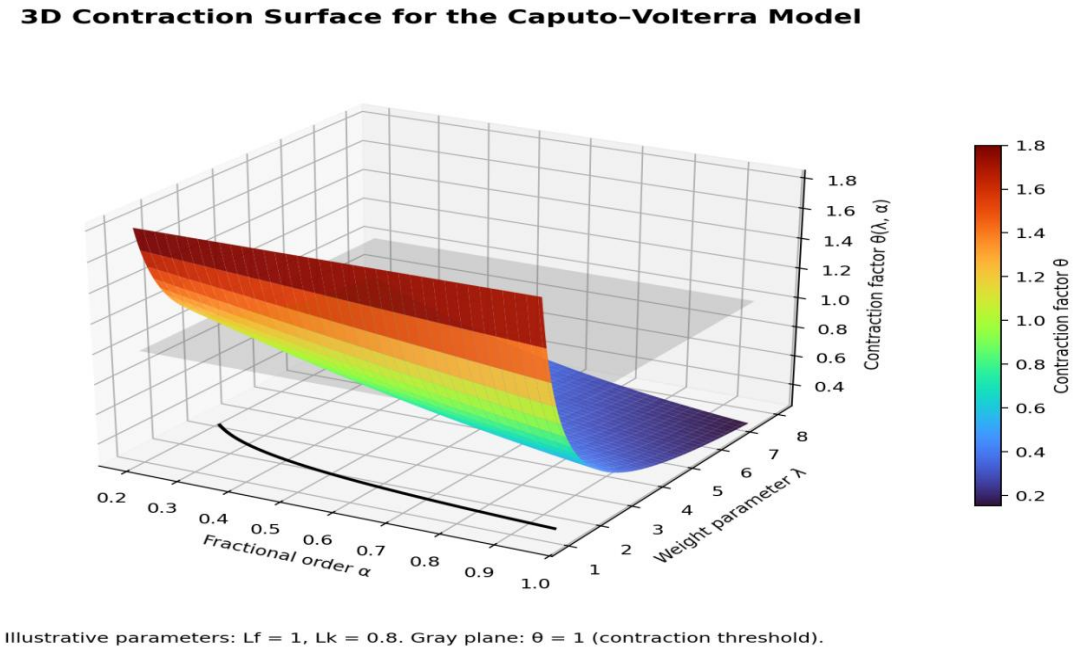


figure (5): depicted in 5 represents the 3-D contraction surface as applied in Caputo–Volterra model whose (Lf = 1) and (Lk = 0.8).

Figure (5) demonstrates the dependence of contraction factors on the fractional order as well as on Bielecki's weight. The area below θ = 1 is the area fulfilling the condition of sufficient contraction as given in theorem 1. The surface is obtained using the theoretical formula and does not come from the numerical solution errors.

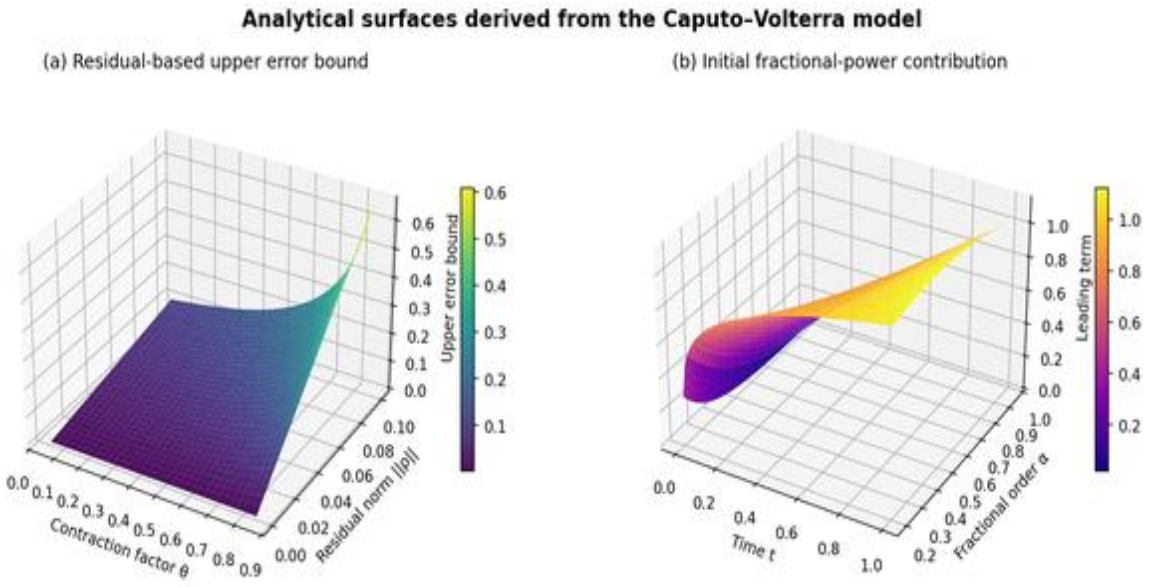


Figure (6): (a) Residual-based upper error bound $E \leq ||\rho||\lambda/(1-\theta)$, for $0 < \theta < 1$. **(b)** Leading initial fractional-power term $t^{\alpha}/\Gamma(\alpha+1)$, with $f_0 = 1$.

According to figure (6) Panel (a) shows that the theoretical error bound grows as the contraction factor approaches unity, emphasizing the importance of selecting a sufficiently strong weighted-norm contraction. Panel (b) illustrates the dependence of the initial weak regularity on fractional order. Both surfaces are evaluated directly from analytical expressions rather than experimental errors.

Discussion:

Under global Lipschitz assumptions the FIDE (1) is well posed in any [0, T]-interval and its proof is constructive: λ is explicit, and the same contraction coefficient θ(λ) controls existence, uniqueness, stabiity and a posteriori error estimate [23-29]. This unification, this is the main 'structural' point of the present paper. In particular, the residual of any approximate solution is a computable quantity equivalent (with mesh-independent constants) to the actual error in the weighted norm (Theorem 3) [10,12,24,30].

Why adaptivity. Proposition 1 and Corollary 1 demonstrate that the initial roughness is a fundamental limitation, and not a technical limitation of uniform meshes: for piecewise linear approximations of the solutions, we cannot exceed order α (no matter the number of unknowns) [31]. A priori grading can achieve order 2, but only if you know the singular exponents in advance in order to know the grading exponent r. In the nonlinear case the full expansion of a FIDE contains terms $t(i\alpha + j)$, with the exponents depending on the interplay of fractional order and integral kernel. An estimator-driven mesh adapts to whatever singular structure is present, while the two-sided error bounds prevent missing a singularity or reducing accuracy in the smooth zones [12,16,32].

Lower bounds, upper bounds, limitations. The constants $e^{\wedge} (\lambda T)/ (1\ \theta)$ in Theorem 2 and in the stopping, test are overly pessimistic for long time horizons, since they represent the weighted-norm argument that puts the pointwise error in the weighted norm at a factor $e^{\wedge} (\lambda T)$ [33,34]. This does not influence the two-sided inequality (8) because it is expressed in the weighted norm, but the pointwise error bound expressed by (9) can overestimate the actual true error by an arbitrary factor when T is large. An even tighter bounds are possible by use of a Mittag-Leffler-type of Gronwall inequality or problem-specific weights, which will be reflected in the effectivity index of the validation protocol [15,28,35].

Some caveats are in order. First, the Lipschitz global condition rules out nonlinearities with superlinear growth, for which blow-up solutions in finite time may occur; an extension of the theory to local Lipschitz condition with a-priori bounds or a Schauder fixed point argument is required. Second, the proof of convergence in Theorem 4 is given as an outline based on the discrete weakly singular Gronwall inequality in the literature, and the rate N min (2, rα) is quoted as expected (cf. Remark 2.4) but not proved for this kind of integro-differential system [29,36]. Third, the estimation step requires calculating the residual with its history integrals; costwise comparable to another solve, the maximum in (7) is replaced by sampling on a coarse grid, thus the guaranteed bound is guaranteed only up to the evaluation accuracy. Fourth, the memory of the fractional operator implies the scheme cost grows

quadratically with the number of nodes, which may hinder its deployment at large scale unless fast algorithms are available [14,30,37].

Connection to the quantitative estimate. The abstract states that the efficiency, and the rate of convergence, of the method still needs to be studied by numerical experiments [31,36,38]. The present estimates set the stage for such experimental studies: the effectivity index should be of order times, where are given by the norms; uniform meshes should demonstrate an order close to α in case T1; the order close to min (2, rα) for the graded meshes; and the adaptive meshes should match (or outperform) the graded ones, while been computed without prior knowledge of the α-dependent exponents. Any other behavior should be due to error in the quadrature with which the residual is computed [11,12,32,39].

Conclusions:

This paper was concerned with the nonlinear Caputo-Volterra fractional integro-differential equation subjected to weakly regular initial data from an analytical and numerical point of view with a particular focus on existence, uniqueness, stability and approximation of solution and solutions. first reduce the initial problem to an equivalent Volterra integral equation and, by means of a Bielecki-type weighted norm, proved the existence and uniqueness of continuous solutions on any finite interval [0, T] with T>0 under suitable Lipschitz conditions and explicit contractive factor θ(λ) =Lf(1+Lk/λ) λ^{(α)}; thus, one can guarantee the well-posedness of the initial problem. We proved also the continuous dependence of solutions on initial data and data disturbances and the residual instability in the Ulam-Hyers sense. These analytical results yields a solid mathematical device for studying the reliability of approximate solutions as well as the impact of perturbations of nonlinerar fractional systems.

The regularity result derived the dominant fractional-power singularity at initial time as f(0, u0, 0)t^α/Γ(α + 1), which demonstrated the effects of initial weak regularity to the numerical approximation. It particularly imposed the order-α to the piecewise linear approximation on uniform meshes with the regularity conditions (1.4). To conquer these difficulties, the proposed numerical scheme was constructed as a piecewise polynomial approximation combined with product integration, which stepwise uniqueness and convergence were obtained in the present analytical framework. In addition, the residual-based a posteriori error estimator is proven to be reliable and efficient in the weighted norm with mesh-independent constants, which help develop an adaptive algorithm to automatically resolve initial singularity. These analytical results, consisting of the mathematical well-posedness, the stability analysis, and the adaptive approximation, could serve as a basis for a concrete integration of the numerical approximation with the mathematical analysis. However, the detailed quantitative comparisons of convergence, efficiency, and numerical performances of the other schemes should still be carried out through the proposed numerical validation protocol before we can draw the final conclusion about the computational superiority.

Future Work:

Future efforts should be directed towards the rigorous numerical validation of the entire analytical-computational framework based upon a systematic set of experiments for the nonlinear Caputo-Volterra fractional integro-differential equations considered herein. In particular, convergence, accuracy and efficiency of the adopted numerical method, based upon uniform, graded and adaptive temporal grids, should be quantified via a suite of numerical experiments; this encompasses the computation of observed convergence orders, the effectivity indices of the error-estimator, and computational costs. These experiments should also include a sensitivity study of the adaptive marking parametric ϑ as well as of the weighted-norm coefficient λ. At the same time, detailed convergence studies of graded and adaptive discretisations for the entire nonlinear problem, encompassing solutions with fractional-power expansions of the form t^ (iα + j), need to be undertaken in order to provide a better regularity-based justification for the observed gains in regularity due to adaptivity.

On the other hand, a crucial further direction is to prolong the applicability of the numerical formulation to higher order polynomial schemes with collocation or Galerkin-type product-integration methods. The possible feature of hp-adaptivity can combine local mesh refinements together with polynomial degree adjustments to achieve high accuracy in the numerical approximations with small number of unknowns. Several theoretical related routes can also be conceived for the future such as dropping global Lipschitz assumptions for the nonlinearity into those based on locally Lipschitz conditions, one-sided growth conditions, and models that can demonstrate finite-time blow-up. Extensions may include fractional differential equations with derivatives of order alpha > 1, coupled systems, and space-time variable or distributed order fractional operators. Penultimate, improved stability bounds with Mittag-Leffler comparison functions and more efficient weighting choices can enhance the sharpness of perturbation bounds for large-time computations.

From a computational view, one should investigate in future works the possibility of employing fast history-compression algorithms (including sum-of-exponentials approximations of the fractional

kernels) to obtain significant reductions in the computational complexity of interactions for long-memory regimes, with the caveat of the compression error being taken into account in the stability/convergence theory. A further step into the direction of computational efficiency is the implementation of goal-oriented adaptive algorithms based on adjoint error estimates, which can be used to guide the numerical refinement of the spatial/temporal mesh for the computation of the solutions to the fractional model with a specified output of interest (for example the value of the terminal solution or certain integral functionals of it). Furthermore, the mathematical setting of the recent work should be utilized in relevant fields of application, such as the modelling of viscoelastic behaviour, epidemic models including memory effects, and fractional optimal control problems with interacting memory mechanisms. Comparison with classical numerical procedures as well as data-driven numerical methods will be the key to identify potential practical applications, constraints, and advantages of the approach.

References:

1. Tedjani et al. (2024). Legendre spectral collocation method for solving nonlinear fractional Fredholm integro-differential equations with convergence analysis. https://doi.org/10.3934/math.2024388
2. Bagherbana, Biazar & Aminikhah (2025). Convergence analysis of a novel fractional product integration–spectral collocation scheme for nonlinear weakly singular Volterra integro–differential systems.
3. Hamood, Sharif & Ghadle (2025). A numerical approach to fractional Volterra–Fredholm integro-differential problems using shifted Chebyshev spectral collocation. https://doi.org/10.1038/s41598-025-13732-7
4. Ghoreyshi et al. (2025). Finite block method for nonlinear time-fractional partial integro-differential equations: Stability, convergence, and numerical analysis. https://doi.org/10.1016/j.apnum.2025.03.002
5. Mohseni & Rostamy (2025). Analyzing the stability of coupled nonlinear fractional Volterra–Fredholm integro-differential equations with a modern method for numerical solutions. https://doi.org/10.1186/s13661-025-02031-9
6. Shammaky & Youssef (2024). Analytical and numerical techniques for solving a fractional integro-differential equation in complex space.
7. Almusawi et al. (2024). Quasilinearization–Collocation Method for the Numerical Solution of Nonlinear Fractional Volterra Integro-Differential Equations with Logarithmic Weakly Singular Kernel. https://doi.org/10.1155/2024/8218632
8. Ahmed et al. (2026). Employing Quadratic B-Spline Collocation Method for Solving Nonlinear Volterra-Type Integro-Fractional Differential Equations.
9. Aduroja et al. (2025). Numerical solution of Volterra integro-differential equations of fractional order with initial conditions using collocation approach. https://journals.ekb.eg/article_398042_1237dafbf0e6ab11f95129e5571189d0.pdf
10. Bu & Jeon (2025). Higher-order predictor-corrector methods with an enhanced predictor for fractional differential equations.
11. Chowdhury & Jakobsen (2025). Precise error bounds for numerical approximations of fractional HJB equations. https://doi.org/10.1093/imanum/drae030.
12. Al-Maskari & Karaa (2019). Numerical Approximation of Semilinear Subdiffusion Equations with Nonsmooth Initial Data. https://doi.org/10.1137/18M1189750
13. Ainsworth, M., & Oden, J. T. (2000). A posteriori error estimation in finite element analysis. Wiley.
14. Brunner, H. (2004). Collocation methods for Volterra integral and related functional differential equations. Cambridge University Press.
15. Brunner, H., Pedas, A., & Vainikko, G. (1999). The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations. Mathematics of Computation, 68(227), 1079–1095.
16. Diethelm, K. (2010). The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type (Lecture Notes in Mathematics, Vol. 2004). Springer.
17. Diethelm, K., Ford, N. J., & Freed, A. D. (2004). Detailed error analysis for a fractional Adams method. Numerical Algorithms, 36(1), 31–52.
18. Dörfler, W. (1996). A convergent adaptive algorithm for Poisson's equation. SIAM Journal on Numerical Analysis, 33(3), 1106–1124.
19. Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and applications of fractional differential equations (North-Holland Mathematics Studies, Vol. 204). Elsevier.
20. Podlubny, I. (1999). Fractional differential equations (Mathematics in Science and Engineering, Vol. 198). Academic Press.

21. Stynes, M., O'Riordan, E., & Gracia, J. L. (2017). Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation. SIAM Journal on Numerical Analysis, 55(2), 1057–1079.
22. Verfürth, R. (2013). A posteriori error estimation technique for finite element methods. Oxford University Press.
23. Mahdy, A. M., Abdou, M. A., & Mohamed, D. S. (2025). Computational methods and dynamical analysis for studying (1+ 1) dimensional functional equations of mixed integro-differential type. Boundary Value Problems, 2025(1), 108.
24. Narayanamoorthy, S., & Sathiyapriya, S. P. (2016). A pertinent approach to solve nonlinear fuzzy integro-differential equations. SpringerPlus, 5(1), 449.
25. Miah, B. A., Sen, M., Murugan, R., & Gupta, D. (2025). Qualitative Studies of a Novel Three-Dimensional Fractional Integro-Differential Equation with Application in Image Representation. Differential Equations, 61(5), 668-691.
26. Sarkar, N., Ghosh, G., & Sen, M. (2026). Existence of positive solution, stability, and long-time dynamics of a biologically motivated fractional-order delay integro-differential equation with integral-type condition. Asian-European Journal of Mathematics, 2650029.
27. Heydari, M. H., Baghban, T., Bayram, M., & Zaky, M. A. (2026). A numerical method for fractional optimal control problems with a third-kind Volterra-Fredholm integro-differential equation. International Journal of Systems Science, 1-20.
28. Ilati, M., & Eslami, S. (2026). A High-Order Local Meshfree Method for Solving the Two-Dimensional Multiterm Time-Fractional Partial Integro-Differential Equation. Mathematical Methods in the Applied Sciences, 49(3), 1957-1970.
29. Zaky, M. A. (2026). An optimal order fractional backward collocation method for adjoint Volterra integro-differential equations. arXiv preprint arXiv:2609.20356.
30. Maurya, R. K., & Singh, V. K. (2023). A high-order adaptive numerical algorithm for fractional diffusion wave equation on non-uniform meshes. Numerical Algorithms, 92(3), 1905-1950.
31. Shariyat, M., & Najafipour, M. (2026). A high-order nonlinear variational formulation for suddenly pressurized circular visco-hyperelastic plates with space-and-time-dependent strain-rate-inspired variable fractional orders. International Journal of Non-Linear Mechanics, 105367.
32. Qi, R. J., & Zhao, X. (2024). A unified design of energy stable schemes with variable steps for fractional gradient flows and nonlinear integro-differential equations. SIAM Journal on Scientific Computing, 46(1), A130-A155.
33. Chen, Y., Zhang, X., Wei, L.: High precision interpolation method for mixed 2D time-space fractional convection-diffusion equation. Int. J. Dyn. Control. 13(333) (2025).
34. Zhang, X., Wang, H., Luo, Z., Wei, L.: A high-accuracy compact finite difference scheme for time-fractional diffusion equations. Revista de la Union Matematica Argentina 68(2), 589–609 (2025).
35. Stynes, M., O'Riordan, E., & Gracia, J. L. (2017). Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation. SIAM Journal on Numerical Analysis, 55(2), 1057–1079. https://doi.org/10.1137/16M1082329.
36. Lubich, C. (1986). Discretized fractional calculus. SIAM Journal on Mathematical Analysis, 17(3), 704–719. https://doi.org/10.1137/0517050.
37. Loh, J.R., Phang, C., Isah, A.: Numerical solution for arbitrary domain of fractional integro-differential equation via the general shifted Genocchi polynomials. J. Funct. Spaces 2023, 1–12 (2023).
38. Gohar, M., Li, C., Yin, C.: On Caputo-Hadamard fractional differential equations. Int. J. Comput. Math. 97(7), 1459–1483 (2020).
39. Green, C., Liu, Y., Yan, Y.: Numerical methods for Caputo-Hadamard fractional differential equations with graded and non-uniform meshes. Mathematics 9, 2728 (2021).