



Rolle's Theorem and Its Generalization Via the Katugampola Fractional Derivative

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مبرهنة رول وتعميمها باستخدام مشتقة كاتو غامبول الكسرية

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Abstract:

Fractional calculus has extended the classical notions of differentiation and integration to non-integer orders and has become an important tool in many areas of mathematics and applied sciences. Among the modern formulations of fractional operator, The Katugampola fractional derivative provides a unified framework that generalizes several classical fractional derivatives, including the Riemann-Liouville and Hadamard types. In this paper, we review the definitions and some fundamental properties of The Katugampola fractional derivative. We also revisit fractional versions of classical results from differential calculus, particularly Rolle's theorem and the Mean Value Theorem, within the framework of The Katugampola fractional operator. Furthermore, we present and discuss a general form of Rolle's theorem in this context. An illustrative example is provided to demonstrate the applicability of this result.

Keywords: Rolle's theorem, Katugampola Fractional, Fractional Derivatives, Mean Value Theorem.

الملخص:

يمتد حساب التفاضل والتكامل الكسري ليشمل المفاهيم الكلاسيكية للاشتقاق والتكامل إلى رتب غير صحيحة، وقد أصبح أداة مهمة في العديد من مجالات الرياضيات والعلوم التطبيقية. ومن بين الصياغات الحديثة للمؤثرات الكسرية، يوفر المشتق الكسري لكاتو غامبول إطاراً موحداً يعمم عدة مشتقات كسرية كلاسيكية، بما في ذلك نوعي ريمان-ليوفيل وهادامارد. في هذا البحث، نستعرض تعريف وبعض الخصائص الأساسية للمشتق الكسري لكاتو غامبول. كما نعيد تناول الصيغ الكسرية لبعض النتائج الكلاسيكية في حساب التفاضل، ولا سيما مبرهنة رول ومبرهنة القيمة المتوسطة، وذلك ضمن إطار مؤثر

كاتوغامبولا الكسري. علاوة على ذلك، نقدم ونناقش صيغة عامة لمبرهنة رول في هذا السياق. ويتم تقديم مثال توضيحي لبيان قابلية تطبيق هذه النتيجة.

الكلمات المفتاحية: مبرهنة رول، مشتقة كاتوغامبولا، التفاضل الكسري، مبرهنة القيمة المتوسطة.

Introduction:

Fractional calculus is a rapidly developing branch of mathematics analysis that extends the classical concepts of differentiation and integration to non-integer orders. This generalization has proven useful in modeling complex phenomena across various scientific and engineering disciplines, where standard integer-order calculus may fail to capture memory effects or hereditary properties of dynamic systems. Over the last few decades, numerous definitions of fractional derivatives have been proposed, each with distinct properties and applications [1-4].

Among these, The Katugampola fractional derivatives, introduced by Udit N. Katugampola in 2011 [5-9], has attracted significant attention due to its ability to unify several classical fractional derivatives, including the Riemann-Liouville and Hadamard derivatives. This operator offers a flexible framework that preserves many important analytical properties, making it suitable for both theoretical investigations and applied problems.

Classical results from differential calculus, such as Rolle's theorem and the Mean Value Theorem, play a central role in analysis and its applications. Fractional analogues of these theorems have been established in the literature using various fractional operators, including the Katugampola derivatives. These generalizations extend the notion of a derivative to non-integer order while maintaining key properties of classical calculus.

In this paper, we review the definition and fundamental properties of the Katugampola fractional derivative and revisit known fractional versions of Rolle's theorem and the Mean Value Theorem. We also discuss a general form of Rolle's theorem in this fractional context and provide an illustrative example to demonstrate its applicability. Our goal is to clarify the theoretical framework and highlight the relevance of these results for further study in fractional calculus.

Types of Fractional Derivatives:

1. Riemann-Liouville Fractional Derivative [4].

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-1)} \frac{d^n}{dx^n} \int_0^{\infty} (x-t)^{n-\alpha-1} f(t) dt, n-1 < \alpha < n$$

Remark:

One can easily see that the above definition doesn't give zero for constant functions.

2. Caputo Fractional Derivative [7].

$$D^{\alpha}f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{n-\alpha-1}} dt, & n-1 < \alpha < n \\ \frac{d^n f(x)}{dx^n}, & \alpha = n \end{cases}$$

The above definition doesn't satisfy many standard properties such as the rule of multiplication, and the chain rule, so Khalil and others introduced the concept of conformable fractional derivative.

3. Definition [10][11]. If $f: [0, \infty) \rightarrow \mathbb{R}$, the conformable fractional derivative to f of order α is defined as follows:

$$D^{\alpha}f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad \forall t > 0, \alpha \in (0,1)$$

Katugampola extended Khalil's definition and then named after him.

4. Definition [6][3]. Let $\alpha \in (n, n+1)$, $n \in \mathbb{N}$, and f is a differentiable function on its domain and $\alpha > 0$.

he fractional derivative of order α is defined as follows:

$$D^{\alpha}f(t) = \lim_{\epsilon \rightarrow 0} \frac{f^{(n)}(te^{\epsilon t^{n-\alpha}}) - f^{(n)}(t)}{\epsilon}$$

$\forall t > 0, \alpha \in (n, n+1)$, if the limit exists.

Theorem [4] [6]: If $f: (a, +\infty) \rightarrow \mathbb{R}$ is a Katugampola fractional function and differentiable at $x = a > 0$, with $\alpha \in (0,1)$, then f is continuous at $x = a$.

Proof. If $f: [0, \infty) \rightarrow \mathbb{R}$ we define the applied fractional derivative f from the order α as follows;

$$D^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \forall t > 0, \alpha \in (0,1)$$

If f is differentiable of α times for some $\alpha \in (0, a)$, $a > 0$ and $\lim_{t \rightarrow 0} f^\alpha(t)$ **exists then**:

$$f^\alpha(0) = \lim_{t \rightarrow 0} f^\alpha(t)$$

Therefore, f is continuous at $x = a$.

Theorem [6]: Let $\alpha \in (0,1]$ and let the functions f and g satisfy the conditions for the fractional derivative at $x = a, t > 0$. **Then:**

1. $D^\alpha[af + bg] = aD^\alpha(f) + bD^\alpha(g), \quad a, b \in \mathbb{R}$
2. $D^\alpha(t^n) = nt^{n-\alpha}, \quad n \in \mathbb{R}$
3. $D^\alpha(c) = 0, \quad c \in \mathbb{R}$
4. $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$
5. $D^\alpha\left(\frac{f}{g}\right) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$
6. $D^\alpha(f \circ g) = f'(g(t))g'(t)$
7. $D^\alpha(f) = t^{1-\alpha} \frac{df}{dt}(t)$

Proof:

$$\begin{aligned} 1. \quad D^\alpha[af + bg] &= \lim_{\varepsilon \rightarrow 0} \frac{(af+bg)(te^{\varepsilon t^{-\alpha}}) - (af+bg)(t)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{af(te^{\varepsilon t^{-\alpha}}) + bg(te^{\varepsilon t^{-\alpha}}) - af(t) - bg(t)}{\varepsilon} \\ &= a \cdot \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon} + b \cdot \lim_{\varepsilon \rightarrow 0} \frac{g(te^{\varepsilon t^{-\alpha}}) - g(t)}{\varepsilon} = aD^\alpha(f) + bD^\alpha(g) \end{aligned}$$

$$\begin{aligned} 2. \quad D^\alpha(t^n) &= \lim_{\varepsilon \rightarrow 0} \frac{t^n e^{n\varepsilon t^{-\alpha}} - t^n}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{(te^{n\varepsilon t^{-\alpha}})^n - t^n}{\varepsilon} \\ &= t^n \lim_{\varepsilon \rightarrow 0} \frac{e^{n\varepsilon t^{-\alpha}} - 1}{\varepsilon} = t^n \lim_{\varepsilon \rightarrow 0} \frac{\left[1 + n\varepsilon t^{-\alpha} + \frac{n^2 \varepsilon^2 t^{-2\alpha}}{2!} + \dots\right] - 1}{\varepsilon} \\ &= t^n \lim_{\varepsilon \rightarrow 0} \frac{\left[n\varepsilon t^{-\alpha} + \frac{n^2 \varepsilon^2 t^{-2\alpha}}{2!} + \dots\right]}{\varepsilon} = t^n \lim_{\varepsilon \rightarrow 0} \left[nt^{-\alpha} + \frac{n^2 \varepsilon t^{-2\alpha}}{2!} + \dots\right] \\ &= t^n nt^{-\alpha} = nt^{n-\alpha} \end{aligned}$$

$$3. \quad D^\alpha(c) = \lim_{\varepsilon \rightarrow 0} \frac{c - c}{\varepsilon} = 0$$

$$\begin{aligned} 4. \quad D^\alpha(fg) &= \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}})g(te^{\varepsilon t^{-\alpha}}) - f(t)g(t)}{\varepsilon} = \\ &= \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}})g(te^{\varepsilon t^{-\alpha}}) - f(t)g(te^{\varepsilon t^{-\alpha}}) + f(t)g(te^{\varepsilon t^{-\alpha}}) - f(t)g(t)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} g(te^{\varepsilon t^{-\alpha}}) \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon} + \lim_{\varepsilon \rightarrow 0} f(te^{\varepsilon t^{-\alpha}}) \lim_{\varepsilon \rightarrow 0} \frac{g(te^{\varepsilon t^{-\alpha}}) - g(t)}{\varepsilon} = \\ &= g(t) \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon} + f(t) \lim_{\varepsilon \rightarrow 0} \frac{g(te^{\varepsilon t^{-\alpha}}) - g(t)}{\varepsilon} = f(t)D^\alpha(g) + g(t)D^\alpha(f) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{g(t)f(te^{\varepsilon t^{-\alpha}}) - f(t)g(te^{\varepsilon t^{-\alpha}})}{\varepsilon g(t)g(te^{\varepsilon t^{-\alpha}})} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{g(t)f(te^{\varepsilon t^{-\alpha}}) - f(t)g(t) + f(t)g(t) - f(t)g(te^{\varepsilon t^{-\alpha}})}{\varepsilon g(t)g(te^{\varepsilon t^{-\alpha}})} \\ &= \left(g(t) \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon} - f(t) \lim_{\varepsilon \rightarrow 0} \frac{g(te^{\varepsilon t^{-\alpha}}) - g(t)}{\varepsilon} \right) \lim_{\varepsilon \rightarrow 0} \frac{1}{g(t)g(te^{\varepsilon t^{-\alpha}})} \\ &= \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2} \end{aligned}$$

The sixth item is proven in two different ways:

First Method:

Assume that $u = g(t)$ is differentiable at the point $a > 0$, and $y = f(y)$ is differentiable at the point $b = g(a) > 0$. Let $\varepsilon > 0$, and $\Delta y = f(b e^{\varepsilon b^{-\alpha}}) - f(b)$, since $b e^{\varepsilon b^{-\alpha}} = b + \varepsilon b^{1-\alpha} + O(\varepsilon^2) = b + \Delta u$, where $\Delta u = \varepsilon b^{1-\alpha} + O(\varepsilon^2)$

We have the equation:

$$\Delta y = f(b e^{\varepsilon b^{-\alpha}}) - f(b) = \frac{f(b e^{\varepsilon b^{-\alpha}}) - f(b)}{\Delta u} \Delta u = D^1 f(b) \Delta u + \varepsilon_1 \Delta u \quad (1)$$

where $\varepsilon_1, \Delta u \rightarrow 0$. If we set ε_1 to 0, then ε_1 is a continuous function of Δu .

Now, if Δt , is the increasing rate $t, \Delta u, \Delta y$ respectively (with the possibility that some may be zero), then using equation (1), **we can write:**

$$\begin{aligned} \Delta u &= D^\alpha g(a) \Delta t + \varepsilon_2 \Delta t \quad \text{where } \varepsilon_2 \rightarrow 0 \text{ as } \Delta t \rightarrow 0 \\ \Delta y &= [D^\alpha f(b) + \varepsilon_1] \Delta u = [D^\alpha f(b) + \varepsilon_1][D^\alpha g(a) + \varepsilon_2] \Delta t \quad (2) \end{aligned}$$

Now, taking $\Delta t = \varepsilon$ and $\Delta t \rightarrow 0$, we have the fractional derivative **of the composite function:**

$$\begin{aligned} D^\alpha (f \circ g)(a) &= \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} = \lim_{\varepsilon \rightarrow 0} [D^\alpha f(b) + \varepsilon_1][D^\alpha g(a) + \varepsilon_2] \\ &= D^\alpha f(b) D^\alpha g(a) = f^\alpha(g(a)) D^\alpha g(a) \quad (3) \end{aligned}$$

Note that equation (2) can be written in the form of equation (1), and it is necessary to use the derivative α in equation (2).

Second Method:

We derive the result by following a standard limit approach. To achieve this goal, on one hand, if the function g is constant in an interval containing a , then $D^\alpha (f \circ g)(a) = 0$. On the other hand, assume that g is not constant in the neighborhood of a . In this case, we can find $\varepsilon_0 > 0$ such that:

$$x_1, x_2 \in (a - \varepsilon_0, a + \varepsilon_0), g(x_1) \neq g(x_2).$$

Now, since g is continuous at a , for every $\varepsilon > 0$:

$$\begin{aligned} D^\alpha (f \circ g)(t) &= \lim_{\varepsilon \rightarrow 0} \frac{f(g(a e^{\varepsilon a^{-\alpha}})) - f(g(t))}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{f(g(a e^{\varepsilon a^{-\alpha}})) - f(g(a))}{g(a e^{\varepsilon a^{-\alpha}}) - g(a)} \cdot \frac{g(a e^{\varepsilon a^{-\alpha}}) - g(a)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{f(g(a) + \varepsilon_1) - f(g(a))}{\varepsilon_1} \cdot \frac{g(a e^{\varepsilon a^{-\alpha}}) - g(a)}{\varepsilon}, \quad \varepsilon_1, \varepsilon > 0 \\ &= \lim_{\varepsilon_1 \rightarrow 0} \frac{f(g(a) + \varepsilon_1) - f(g(a))}{\varepsilon_1} \cdot \lim_{\varepsilon \rightarrow 0} \frac{g(a e^{\varepsilon a^{-\alpha}}) - g(a)}{\varepsilon} \\ &= f'(g(a)) D^\alpha g(a), \quad a > 0 \end{aligned}$$

This is the basis for the chain rule:

$$\begin{aligned} D^\alpha f(t) &= \lim_{\varepsilon \rightarrow 0} \frac{f(t e^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{f\left(t + \varepsilon t^{1-\alpha} + \frac{\varepsilon^2 t^{2-\alpha}}{2!} + \dots\right) - f(t)}{\varepsilon} \end{aligned}$$

Take $h = \varepsilon t^{1-\alpha} + \frac{\varepsilon^2 t^{2-\alpha}}{2!} + \dots = \varepsilon t^{1-\alpha} \left[1 + \frac{\varepsilon t}{2!} + \dots\right]$, Where $h, \varepsilon \rightarrow 0$.

Then:

$$\begin{aligned} D^\alpha f(t) &= \lim_{\varepsilon \rightarrow 0} \frac{\frac{f(t+h) - f(t)}{h}}{t^{1-\alpha} \left[1 + \frac{\varepsilon t}{2!} + \dots\right]} = \lim_{\varepsilon \rightarrow 0} \frac{\frac{f(t+h) - f(t)}{h}}{t^{1-\alpha}} \\ &= t^{1-\alpha} \cdot \lim_{\varepsilon \rightarrow 0} \frac{f(t+h) - f(t)}{h} = t^{1-\alpha} \cdot \frac{df(t)}{dt} \end{aligned}$$

Corollary. Let $a, n \in \mathbb{R}, \alpha \in (0,1)$ then the following holds:

1. $D^\alpha (t^n) = n t^{n-\alpha}, \quad n \in \mathbb{R}$
2. $D^\alpha (c) = 0, \quad c \in \mathbb{R}$
3. $D^\alpha (e^{ax}) = a x^{1-\alpha} e^{ax}, \quad a \in \mathbb{R}$
4. $D^\alpha (\sin bx) = b x^{1-\alpha} \cos bx, \quad b \in \mathbb{R}$
5. $D^\alpha (\cos bx) = -b x^{1-\alpha} \sin bx, \quad b \in \mathbb{R}$

$$6. \quad D^\alpha \left(\frac{1}{\alpha} t^\alpha \right) = 1$$

Corollary. Let $t > 0, \alpha \in (0,1]$, then the following holds:

1. $D^\alpha \left(\sin \frac{1}{\alpha} t^\alpha \right) = \cos \frac{1}{\alpha} t^\alpha$
2. $D^\alpha \left(\cos \frac{1}{\alpha} t^\alpha \right) = -\sin \frac{1}{\alpha} t^\alpha$
3. $D^\alpha \left(e^{\frac{1}{\alpha} t^\alpha} \right) = e^{\frac{1}{\alpha} t^\alpha}$

$$\begin{aligned} f(ae^{ax}) - f(a) &= \frac{f(ae^{ax}) - f(a)}{\varepsilon} \cdot \lim_{\varepsilon \rightarrow 0} \varepsilon \\ &\because \lim_{a \rightarrow a^+} [f(ae^{ax}) - f(a)] = \lim_{a \rightarrow a^+} \frac{f(ae^{ax}) - f(a)}{\varepsilon} \cdot \lim_{\varepsilon \rightarrow 0} \varepsilon \\ ae^{ax} &= a \left[1 + \alpha a^{-a} + \frac{\varepsilon^2 a^{-2a}}{2!} + \dots \right] = a + \alpha a^{-a} + \frac{\varepsilon^2 a^{-2a}}{2!} + \dots = a + \alpha a^{-a} + O(\varepsilon^2) \\ &\therefore \text{shiling, } h = \alpha a^{-a} + O(\varepsilon^2), a + h = ae^{ax} \\ \lim_{a \rightarrow a^+} [f(a+h) - f(a)] &= \lim_{a \rightarrow a^+} \frac{f(ae^{ax}) - f(a)}{\varepsilon} \cdot \lim_{\varepsilon \rightarrow 0} \varepsilon = D^\alpha f(a), 0 = 0 \end{aligned}$$

Thus, $\lim_{a \rightarrow a^+} f(a+h) = f(a)$, and therefore it is continuous at a .

Theorem [6]. Assume that f, g are differentiable at the point a , then:

1. $D^\alpha [af + bg] = aD^\alpha(f) + bD^\alpha(g), \quad a, b \in \mathbb{R}$
2. $D^\alpha(r^n) = nr^{-\alpha}, \quad n \in \mathbb{R}$
3. $D^\alpha(c) = 0, \quad c \in \mathbb{R}$
4. $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$
5. $D^\alpha \left(\frac{f}{g} \right) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$
6. $D^\alpha(f \circ g) = f'(g(t))g'(t)$
7. $D^\alpha(f) = t^{-\alpha} \frac{df}{dt}(t)$

Rolle's theorem and mean value theorem for fractional Derivative functions of Type of Katugampola:

1. **Rolle's theorem** [9]. Let $d > 0, f: [a, b] \rightarrow \mathbb{R}$ such that

(i) f is continuous on $[a, b]$.

(ii) f at α is differentiable on (a, b) , and for some $\alpha \in (0,1)$, $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that $D^\alpha(f)(c) = 0$.

Example1. $f(x) = x^3 - x$ satisfies Roll's theorem for fractional differentiable functions of Katugampola type on the interval $[-1,1]$.

Because $\alpha \in (0,1]$, $D^\alpha f(x) = 3x^{3-\alpha} - x^{1-\alpha}$.

The map is differentiable, so it is continuous on $\mathbb{R}$ and this implies that $f(x)$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$. In addition, $f(-1) = f(1) = 0$.

Thus, the function meets all Rule's theorem conditions on $[-1,1]$ and so there exist

$c \in (-1,1)$ such that $D^\alpha(f)(c) = 0$. Now $D^\alpha(f)(c) = 3c^{3-\alpha} - c^{1-\alpha} = 0$

$$\Rightarrow c = 0 \text{ or } c = \pm \frac{1}{\sqrt{3}}$$

Example2. The map $f(x) = \cos \frac{1}{\alpha} t^\alpha, \alpha \in (0,1]$ satisfies Rolle's theorem conditions for fractional derivative of Katugampola type hold's on the interval $[-\alpha(\frac{\pi}{2})^{\frac{1}{\alpha}}, \alpha(\frac{\pi}{2})^{\frac{1}{\alpha}}]$.

Because: $D^\alpha \left(\cos \frac{1}{\alpha} t^\alpha \right) = -\sin \frac{1}{\alpha} t^\alpha, \alpha \in (0,1]$

$\Rightarrow f(x)$ is differentiable $\Rightarrow f(x)$ is continuous on $\mathbb{R} \Rightarrow f(x)$ is continuous on the interval $[-\alpha(\frac{\pi}{2})^{\frac{1}{\alpha}}, \alpha(\frac{\pi}{2})^{\frac{1}{\alpha}}]$

and differentiable on the interval $(-\alpha(\frac{\pi}{2})^{\frac{1}{\alpha}}, \alpha(\frac{\pi}{2})^{\frac{1}{\alpha}})$ and $f(-\alpha(\frac{\pi}{2})^{\frac{1}{\alpha}}) = f(\alpha(\frac{\pi}{2})^{\frac{1}{\alpha}})$

Thus f meets all Rolle's theorem conditions on our interval and so there exists:

$$c \in \left(-\alpha(\frac{\pi}{2})^{\frac{1}{\alpha}}, \alpha(\frac{\pi}{2})^{\frac{1}{\alpha}} \right)$$

such that $D^\alpha f(c) = 0 \Rightarrow D^\alpha f(x) = -\sin \frac{1}{\alpha} t^\alpha = 0$

$$\Rightarrow \frac{1}{\alpha} t^\alpha = 0 \Rightarrow t = 0$$

The mean value theorem for fractional differentiable maps:

Theorem [6]. Let $\alpha > 0, f: [a, b] \rightarrow \mathbb{R}$ satisfies:

1. f is continuous on $[a, b]$,

2. f is differentiable at α on (a, b) ,
for some $\alpha \in (0, 1)$, then there exists $c \in (a, b)$ **such that:**

$$D^\alpha(f)(c) = \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}}$$

Proof. Let $g(x) = f(x) - y$ with gradient $\frac{f(b)-f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}}$

$$\text{Then: } y - f(a) = \frac{f(b)-f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}} \cdot \left(\frac{x^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}\right)$$

$$\Rightarrow y = f(a) + \frac{f(b)-f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}} \cdot \left(\frac{x^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}\right)$$

$$\text{So: } g(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}} \cdot \left(\frac{x^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}\right)\right)$$

Now, we apply Rolle's Conditions on $g(x)$ over $[a, b]$.

1. $g(x)$ is continuous over $[a, b]$.
2. $g(x)$ is differentiable over $[a, b]$ because

$$D^x(g)(x) = D^\alpha(f)(x) - \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}}.$$

$$3. g(a) = f(a) - f(a) + \frac{f(b)-f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}} \cdot \left(\frac{a^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}\right) = 0$$

$$4. g(b) = f(b) - f(a) - \frac{f(b)-f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}} \cdot \left(\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}\right) = 0$$

Therefore, there exists $c \in (a, b)$ **such that:**

$$D^\alpha(g)(c) = 0. D^\alpha(f)(c) - \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}} = 0$$

$$D^\alpha(f)(c) = \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}}.$$

Example (1): the map $f(x) = 2x^{\frac{3}{2}} - 4\sqrt{x} + 1$ satisfies the mean value theorem on $[1, 4]$ for fractional differentiable maps.

Because: $\alpha \in (0, 1)$, $D^\alpha f(x) = 3x^{\frac{3}{2}-\alpha} - 2x^{0.5-\alpha}$. The map f is differentiable and so it is continuous on $\mathbb{R}$ so, f is continuous on $[1, 4]$ and differentiable on $(1, 4)$. This implies that there exists $c \in (1, 4)$ **such that:**

$$D^\alpha(f)(c) = \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}} = \frac{9 + 1}{\frac{4^\alpha}{\alpha} - \frac{1^\alpha}{\alpha}}$$

$$\text{Take } \alpha = \frac{1}{2} \Rightarrow 3c - 2 = 5 \Rightarrow c = \frac{7}{3} \in (1, 4).$$

Definition [6]. Let $\alpha \in (n, n + 1]$ for some $n \in \mathbb{N}$, and f is n -differentiable and $t > 0$. Then:

$$D^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f^{(n)}(te^{\varepsilon t^{n-\alpha}}) - f^{(n)}(t)}{\varepsilon}$$

Provided the limit exists.

Theorem [6]. Let $\alpha \in (n, n + 1]$ for some $n \in \mathbb{N}$ and f is $(n+1)$ -differentiable and $t > 0$. Then $D^\alpha f(t) = t^{n+1-\alpha} f^{(n+1)}(t)$.

General Rolle's theorem for fractional derivative.

Let $f: [a, b] \rightarrow \mathbb{R}$ be a map. Satisfies the following:

1. $f^{(n-1)}$ is continuous on $[a, b]$
2. $D^\alpha f$ exists on $(a, b]$ for some $\alpha \in (n, n + 1]$, where α is a fractional derivative of katugampola's type.
3. $f^{(n)}(a) = f^{(n)}(b)$.

Then there exists c such that $c \in (a, b)$ and $D^\alpha f(c) = 0$.

Proof. Let $\beta = \alpha - n$, $g(x) = f^{(n)}(x)$, and f is n -differentiable. **Then:**

$$D^\alpha f(x) = t^{n+1-\alpha} f^{(n+1)}(x) = t^{1-\alpha+n} f^{(n+1)}(x) = t^{1-\beta} f^{(n+1)}(x)$$

$$= t^{1-\beta} \frac{d}{dx} f^{(n)}(x) = t^{1-\beta} g'(x) = D^\beta g(x).$$

Now:

- 1) $g(x)$ is continuous on $[a, b]$
- 2) g is β -differentiable on (a, b) , $\beta \in (0,1)$.
- 3) $g(a) = g(b)$.

So, there exists $c \in (a, b)$ such that $D^\beta g(c)=0$ and this implies $D^\alpha f(c) = 0$.

Example: We show that $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2$ satisfies the general Rolle's theorem conditions on $[0, 1]$, where $\alpha = 1.5$.

Since $D^{1.5}f(x) = \frac{1}{3}3x^{3-1.5} - \frac{1}{2}2x^{2-1.5}$, so f is differentiable on $(0,1)$ for some $\epsilon \in (1,2)$ and this implies f is continuous on $[0,1]$.

Moreover, $D^1f(0) = D^1f(1) = 0 \Rightarrow$ there exists c such that $D^{1.5}f(c) = 0$.

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