



Application of the Variation of (G'/G) -Expansion Method to the Nonlinear Sixth-Order Ramani Equation

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تطبيق التباين لطريقة مفكوك (G'/G) على معادلة راماني غير الخطية من الرتبة السادسة

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Abstract:

Exact traveling wave solutions of the nonlinear sixth-order Ramani equation are obtained in this paper by means of a variant of the (G'/G) -expansion method. The variant pairs the usual auxiliary function with a Riccati-type function F , which turns the problem into a system of algebraic equations that is solved symbolically. Three sets of parameter values are found, and the resulting solutions are expressed through the hyperbolic function's $\tanh$ and $\coth$. Every solution has been verified by direct substitution into the reduced ordinary differential equation.

Keywords: Variation of (G'/G) -expansion method, sixth-order Ramani equation, traveling wave solutions, hyperbolic function solutions.

المخلص:

يتم في هذا البحث الحصول على حلول دقيقة لموجات متنقلة (traveling wave solutions) خاصة بمعادلة "راماني" (Ramani) غير الخطية من الرتبة السادسة، وذلك باستخدام صيغة معدلة لطريقة التوسع بـ (G'/G) . تجمع هذه الصيغة المعدلة بين الدالة المساعدة المعتادة ودالة من نوع "ريكاتي" (Riccati) يُرمز لها بـ F ، مما يحول المسألة إلى نظام من المعادلات الجبرية التي تُحل بطريقة رمزية. وقد تم تحديد ثلاث مجموعات من قيم المعاملات، وعُبر عن الحلول الناتجة بدلالة الدوال الزائدية ($\tanh$ و $\coth$). كما تم التحقق من صحة كل حل عن طريق التعويض المباشر في المعادلة التفاضلية العادية المختزلة.

الكلمات المفتاحية: التباين لطريقة مفكوك (G'/G) ، معادلة "راماني" من الرتبة السادسة، حلول الموجات المتنقلة، حلول بدلالة الدوال الزائدية.

Introduction:

Nonlinear partial differential equations (NPDEs) describe a wide range of phenomena in physics and engineering, and finding closed-form solutions to them remains a central problem of applied mathematics. A great many analytical techniques have been devised for this purpose [1-19]. Among them, the (G'/G) -expansion method and its modifications are popular because they reduce the differential problem to algebraic manipulations that a computer algebra system can carry out. The variant used in this paper extends the standard ansatz with a second auxiliary function F , which obeys a Riccati equation, so that the trial solution contains both G'/G and F'/F . We apply it to the sixth-order Ramani equation. The paper is organized as follows: the method is outlined in Section 2, the equation is solved in Section 3, and Section 4 contains the conclusions.

Outline of the Variation of (G'/G) -expansion Method:

Consider a nonlinear PDE (NPDE) in $(n + 1)$ independent variables $x_1, x_2, x_3, \dots, x_n$ and t . As in [20-22], it is first reduced to an ordinary differential equation by the travelling wave transformation:

$$u = u(x_1, x_2, \dots, x_n, t) = u(\xi), \quad \xi = \sum_{i=1}^n r_i x_i + Vt, \quad (1)$$

where $r_i (i = 1, 2, \dots, n)$ and V (the wave speed) are constants fixed later. The PDE thereby becomes an ordinary differential equation (ODE) for the function $u(\xi)$, which is integrated once or more whenever possible. Its solution is then sought in the form:

$$u(\xi) = \sum_{i=0}^m a_i (G'/G)^i + \sum_{i=1}^m b_i (G'/G)^{i-1} (F'/F), \quad (2)$$

where $G = G(\xi)$ and $F = F(\xi)$ are required to satisfy the Riccati system:

$$G'(\xi) = -G(\xi)F(\xi), \quad (3)$$

$$F'(\xi) = 1 - F^2(\xi), \quad (4)$$

Primes denote differentiation with respect to ξ , $a_i (i = 0, 1, \dots, m)$, $b_i (i = 1, 2, \dots, m)$, are constants whose values are still unknown.

This Riccati system has the following two families of solutions:

$$G(\xi) = \pm \operatorname{sech}(\xi), \quad F(\xi) = \tanh(\xi), \quad (5)$$

$$G(\xi) = \pm \operatorname{csch}(\xi), \quad F(\xi) = \coth(\xi), \quad (6)$$

The integer m is fixed by balancing the highest-order derivative against the dominant nonlinear term of the ODE. Inserting the ansatz (2) into the ODE, using (3) and (4), and setting the coefficients of all powers of G'/G and F to zero gives an algebraic system for the unknowns a_i , b_i , r_i and V . Once this system is solved, substituting the results and (5) or (6) into (2) yields the travelling wave solutions.

Solving the Sixth-Order Ramani Equation:

The sixth-order Ramani equation studied in [22] has the form:

$$u_{xxxxxx} + 15u_x u_{xxxxx} + 15u_{xx} u_{xxxx} + 45u_x^2 u_{xxx} - 5u_{xxx} u_{xx} - 15u_x u_{xt} - 15u_t u_{xx} - 5u_{tt} = 0 \quad (7)$$

In order to look for travelling wave solutions of the Ramani equation (7), we set:

$$u(x, t) = u(\xi), \quad \xi = x - ct$$

With this substitution, (7) reduces to the ordinary differential equation:

$$U^6 + 15U'U^4 + 15U''U^3 + 45(U')^2U'' + 5cU^4 + 30cU'U'' - 5c^2U'' = 0 \quad (8)$$

Balancing U^6 against $(U')^2U''$ in (8) gives $m = 1$.

For $m = 1$, the ansatz (2.2) gives the following form for the solution of (8),

$$u(\xi) = a_0 - a_1 F + b_1 (F^{-1} - F). \quad (9)$$

where a_0, a_1, b_1 are constants to be determined.

Substituting (9) into (8) and gathering the coefficients of:

$(F)^i (i = \pm 1, \pm 3, \pm 5, \pm 7)$, and equating each of them to zero, we arrive at a system of algebraic equations in a_1, b_1 and c :

$$F^7: -90(a_1 + b_1)(a_1 + b_1 + 2)(a_1 + b_1 + 4) = 0 \quad (10)$$

$$F^5: 30(a_1 + b_1)(a_1 + b_1 + 2)(9a_1 + 3b_1 - 2c + 28) = 0 \quad (11)$$

$$F^3: -2(a_1 + b_1)(135a_1^2 + 45a_1b_1 - 60ca_1 + 570a_1 - 45b_1^2 - 30cb_1 + 150b_1 - 5c^2 - 100c + 616) = 0 \quad (12)$$

$$F^1: 2(a_1 + b_1)(45a_1^2 - 45a_1b_1 - 30ca_1 + 150a_1 - 45b_1^2 - 90b_1 - 5c^2 - 40c + 136) = 0 \quad (13)$$

$$F^{-7}: 90b_1(b_1 - 2)(b_1 - 4) = 0 \quad (14)$$

$$F^{-5}: 30b_1(b_1 - 2)(6a_1 - 3b_1 - 2c + 28) = 0 \quad (15)$$

$$F^{-3}: 2b_1(45a_1^2 - 135a_1b_1 - 30ca_1 + 420a_1 - 45b_1^2 + 30cb_1 - 150b_1 - 5c^2 - 100c + 616) = 0 \quad (16)$$

$$F^{-1}: -2b_1(45a_1^2 - 45a_1b_1 - 30ca_1 + 240a_1 - 45b_1^2 + 90b_1 - 5c^2 - 40c + 136) = 0 \quad (17)$$

Solving the system (10)–(17) for a_1, b_1 and c with Maple yields **three cases**:

Case (1):

$$a_1 = -2, \quad b_1 = 0, \quad c = 4 \left(\frac{5 \pm \sqrt{45}}{10} \right) \quad (18)$$

Case (2):

$$a_1 = -2, \quad b_1 = 2, \quad c = 4 \left(\frac{5 \pm \sqrt{45}}{10} \right) \quad (19)$$

Case (3):

$$a_1 = -4, \quad b_1 = 2, \quad c = 16 \left(\frac{5 \pm \sqrt{45}}{10} \right) \quad (20)$$

Substituting the cases into (9) and using (5) or (6) gives the solutions below. Cases 1 and 2 lead to the same pair of solutions, so Case 2 is not listed separately. The constant a_0 drops out of (8), so it is arbitrary; it is set to zero below:

According to **Case (1):**

Type (1):

$$u(x, t) = a_0 - a_1 F(\xi) + b_1 (F^{-1} - F(\xi))$$

$$u_1(x, t) = 2 \tanh \left[x - 4 \left(\frac{5 \pm \sqrt{45}}{10} \right) t \right] \quad (21)$$

Type (2):

$$u_2(x, t) = 2 \coth \left[x - 4 \left(\frac{5 \pm \sqrt{45}}{10} \right) t \right] \quad (22)$$

According to **Case (3):**

$$u_3(x, t) = 4 \tanh \left[x - 16 \left(\frac{5 \pm \sqrt{45}}{10} \right) t \right] + 2 \coth \left[x - 16 \left(\frac{5 \pm \sqrt{45}}{10} \right) t \right] - 2 \tanh \left[x - 16 \left(\frac{5 \pm \sqrt{45}}{10} \right) t \right]$$

$$= 2 \tanh \left[x - 16 \left(\frac{5 \pm \sqrt{45}}{10} \right) t \right] + 2 \coth \left[x - 16 \left(\frac{5 \pm \sqrt{45}}{10} \right) t \right] \quad (23)$$

Remarks: (i) Substituting u_1 , u_2 and u_3 into (8) confirms that each of them satisfies the equation (checked symbolically for both signs of c). (ii) Cases 1 and 2 share the same wave speeds and lead to the same pair of solutions, and in Case 3 the identity $\tanh z + \coth z = 2 \coth 2z$ gives $u_3 = 4 \coth[2(x - ct)]$. Hence the method yields three distinct solutions: $u_1 = 2 \tanh \xi$ and $u_2 = 2 \coth \xi$ with $c = 2 \pm 6\sqrt{5}/5$ ($c \approx 4.683$ or -0.683), and u_3 with $c = 8 \pm 24\sqrt{5}/5$ ($c \approx 18.733$ or -2.733). The $\tanh$ solution is a kink-type wave, while the $\coth$ -type solutions are singular at $\xi = 0$.

Conclusion:

A variation of the (G'/G) -expansion method, in which a Riccati-type function F accompanies G'/G , was applied to the sixth-order Ramani equation, and exact travelling wave solutions in terms of $\tanh$ and $\coth$ were obtained. The procedure rests on a polynomial ansatz followed by algebra that is easily automated, so it can also be tried on other nonlinear evolution equations of mathematical physics. Only three distinct solutions were found here (see the remarks above); other choices of the ansatz degree m may give further solutions, which is left for future work.

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